Regions-madogram in spatial processes

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Abstract: Spatial environmental processes often exhibit dependence in their large values. In order to model such processes their dependence properties must be characterized and quantified. In this paper we introduce a regions-madogram, $\nu_F(\mathbf{A},\mathbf{B})$, that evaluates the dependence among extreme observations located in two separated regions, $\mathbf{A}$ and $\mathbf{B}$, of $\mathbb{Z}^2$. We compute the range of this new dependence measure and compare it with extremal coefficients, finding generalizations of the known relations in pairwise approach. The results are illustrated in two max-stable processes: the Schlather's and the Geometric Gaussian models.

1 Introduction

Quantifying dependence is a central issue when leading with spatial observations. In conventional geostatistics, the degree of pairwise spatial dependence in observations from a max-stable random field $\mathbf{X}=\{X_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^2}$ is determined by the variogram

$$\gamma(\mathbf{i},\mathbf{j})=\frac{1}{2}E\left(X_{\mathbf{i}}-X_{\mathbf{j}}\right)^2.$$

If finite, $\gamma(\mathbf{i},\mathbf{j})$ captures some of the spatial structure, capturing it completely if the field is Gaussian. However, it is not possible to ensure that $\gamma(\mathbf{i},\mathbf{j})$ is finite for fields of maxima, so this second-order statistic may not be well-adapted for extremes. To ensure that the moment quantities are finite, Naveau *et al.* (2005) introduced the following type of first-order variograms,

$$\nu_F\left(\mathbf{i},\mathbf{j}\right)=\frac{1}{2}E\left|F(X_{\mathbf{i}})-F(X_{\mathbf{j}})\right|,$$

where F is the common distribution function of $X_{\mathbf{i}},\ \mathbf{i}\in\mathbb{Z}^2$.

More recently, Cooley *et al.* (2006) investigated the basic properties of first-order variograms, also called madograms, for spatial extreme fields. In particular, they derived a few relationships between the

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normalized madogram, ν_F, and the extremal coefficient defined in Schlather (2002) and Schlather and Tawn (2003) as

$$P\left(\max(X_{\mathbf{i}}, X_{\mathbf{j}}) \le u\right) = F^{\varepsilon(\mathbf{i},\mathbf{j})}(u).$$

The normalized madogram fully characterizes the extremal coefficient since we have

$$\varepsilon(\mathbf{i},\mathbf{j}) = \frac{1 + 2\nu_F(\mathbf{i},\mathbf{j})}{1 - 2\nu_F(\mathbf{i},\mathbf{j})}, \tag{1.1}$$

or equivalently

$$\nu_F(\mathbf{i},\mathbf{j}) = \frac{1}{2} - \frac{1}{\varepsilon(\mathbf{i},\mathbf{j}) + 1}. \tag{1.2}$$

Despite that, in practice, the pairwise dependence measures seem to be the most useful and easily understandable, we shall introduce a regions-madogram, that evaluates the dependence among extreme observations located in two separated regions, $\mathbf{A}$ and $\mathbf{B}$, of $\mathbb{Z}^2$.

We illustrate our results with the Geometric Gaussian and the Schlater´s models for random fields.

2 Measuring spatial dependence for maxima over separated regions

We define a normalized madogram to assess dependence among extreme observations located in two separated regions $\mathbf{A}$ and $\mathbf{B}$ of $\mathbb{Z}^2$, as follows.

Definition 2.1 *Let $\mathbf{X}$ be a max-stable random field and $\mathbf{A}$ and $\mathbf{B}$ two separated regions of $\mathbb{Z}^2$. The regions-madogram, $\nu_F(\mathbf{A},\mathbf{B})$, is defined as*

$$\nu_F(\mathbf{A},\mathbf{B}) = \frac{1}{2\left|\mathbf{A}\right|\left|\mathbf{B}\right|} \sum_{\mathbf{i}\in\mathbf{A}} \sum_{\mathbf{j}\in\mathbf{B}} E\left|F(X_{\mathbf{i}}) - F(X_{\mathbf{j}})\right|,$$

or equivalently

$$\nu_F(\mathbf{A},\mathbf{B}) = \frac{1}{\left|\mathbf{A}\right|\left|\mathbf{B}\right|} \sum_{\mathbf{i}\in\mathbf{A}} \sum_{\mathbf{j}\in\mathbf{B}} \nu_F(\mathbf{i},\mathbf{j})$$

Remark 2.1 To measure the dependence among the variables $X_{\mathbf{i}}$ and $X_{\mathbf{j}}, \mathbf{j} \in V^{(n)}(\mathbf{i})$, where

$$V^{(n)}(\mathbf{i}) = \left\{\mathbf{j} \in \mathbb{Z}^2 : \max\{|i_s - j_s| : s = 1, 2\} = n\right\},$$

we can consider the sequence

$$h(n) = \nu_F(\mathbf{i}, n) = \nu_F(\mathbf{i}, V^{(n)}(\mathbf{i})).$$

Remark 2.2 The regions-madogram, $\nu_F(\mathbf{A},\mathbf{B})$, take values in $\left[0, \frac{1}{6}\right]$.

From the definition of regions-madogram we can establish the following properties.

Proposition 2.1 *Suppose that* $\mathbf{X}$ *is a max-stable random field.*

1. *Let* $\mathbf{A} \subseteq \mathbb{Z}^2$ *and* $\mathbf{A}+\mathbf{s} = \{\mathbf{i}+\mathbf{s} : \mathbf{i} \in \mathbf{A}\}$, $\mathbf{s} \in \mathbb{Z}^2$. *If* $\mathbf{X}$ *is stationary, then*

$$\nu_F(\mathbf{A}+\mathbf{s}, \mathbf{B}+\mathbf{s}) = \nu_F(\mathbf{A}, \mathbf{B})$$

.

2. *Let* $\mathbf{A}_i$, $i = 1, \ldots, p$, *be disjoint subsets of* $\mathbb{Z}^2$. *Then*

$$\nu_F\left(\bigcup_{i=1}^{p} \mathbf{A}_i, \mathbf{B}\right) = \sum_{i=1}^{p} \alpha_i \nu(\mathbf{A}_i, \mathbf{B}),$$

with $\alpha_i = \frac{|\mathbf{A}_i|}{|\bigcup_{i=1}^{p} \mathbf{A}_i|}$.
In particular,

$$\nu_F(\mathbf{A}, \mathbf{B}) = \sum_{\mathbf{i} \in \mathbf{A}} \frac{1}{|\mathbf{A}|} \nu_F(\{\mathbf{i}\}, \mathbf{B}).$$

The normalized regions-madogram is related to the pairwise extremal coefficients $\varepsilon(\mathbf{i}, \mathbf{j})$, $\mathbf{i} \in \mathbf{A}$, $\mathbf{j} \in \mathbf{B}$, through

$$\nu_F(\mathbf{A}, \mathbf{B}) = \frac{1}{2} - \frac{1}{|\mathbf{A}|\,|\mathbf{B}|} \sum_{\mathbf{i} \in \mathbf{A}} \sum_{\mathbf{j} \in \mathbf{B}} \frac{1}{\varepsilon(\mathbf{i}, \mathbf{j}) + 1}.$$

In the following, we define a new dependence measure, $\varepsilon^*(\mathbf{A}, \mathbf{B})$, that preserves relation (1.1).

Definition 2.2 *Let* $\mathbf{X}$ *be a max-stable random field and* $\mathbf{A}, \mathbf{B} \subseteq \mathbb{Z}^2$. *The coefficient* $\varepsilon^*(\mathbf{A}, \mathbf{B})$ *is defined as*

$$\varepsilon^*(\mathbf{A}, \mathbf{B}) = \frac{1}{\frac{1}{|\mathbf{A}||\mathbf{B}|} \sum_{\mathbf{i} \in \mathbf{A}} \sum_{\mathbf{j} \in \mathbf{B}} \frac{1}{\varepsilon(\mathbf{i}, \mathbf{j})+1}} - 1. \tag{2.1}$$

We have $\varepsilon^*(\mathbf{A}, \mathbf{B}) = \frac{1+2\nu_F(\mathbf{A}, \mathbf{B})}{1-2\nu_F(\mathbf{A}, \mathbf{B})}$.

Remark 2.3 The coefficient $\varepsilon^*(\mathbf{A}, \mathbf{B})$ take values in $[1, 2]$. When $\varepsilon^*(\mathbf{A}, \mathbf{B})$=1, we have the complete dependence for each pair of variables $(X_\mathbf{i}, X_\mathbf{j})$, $\mathbf{i} \in \mathbf{A}, \mathbf{j} \in \mathbf{B}$, and conversely. For each $\mathbf{i} \in \mathbf{A}$ and $\mathbf{j} \in \mathbf{B}$, the variables $X_\mathbf{i}$ and $X_\mathbf{j}$ are independent if and only if $\varepsilon^*(\mathbf{A}, \mathbf{B}) = 2$.

3 Applications

In this section we illustrate our results in well known max-stable processes: the Schlather's model and the Geometric Gaussian model.

Example 3.1 (Schalather's model) In Schlather (2002), a new class of bivariate marginal distributions is defined, by the extremal Gaussian process, in the following way

$$P\left(X_{\mathbf{i}} \leq x, X_{\mathbf{j}} \leq y\right) = \exp\left[-\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)\left(1+\sqrt{1-\frac{2\left(\rho(\|\mathbf{i}-\mathbf{j}\|)+1\right)xy}{(x+y)^2}}\right)\right], \quad x, y \in \mathbb{R},$$

where $\rho(.)$ is the covariance function of the underlying Gaussian process.

We will consider that $\rho(\|\mathbf{i}-\mathbf{j}\|) = \exp(-(\|\mathbf{i}-\mathbf{j}\|))$.

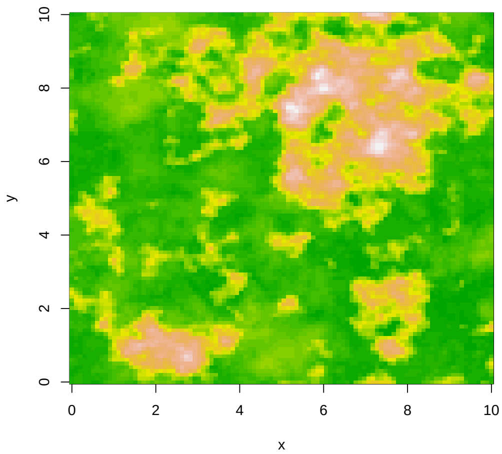

Figure 3.1. Schlather's model with $\rho(\|\mathbf{i}-\mathbf{j}|) = \exp(-(\|\mathbf{i}-\mathbf{j}\|))$

The extremal coefficient is given by

$$\varepsilon(\mathbf{i},\mathbf{j}) = 1+\sqrt{\frac{1-\rho(\|\mathbf{i}-\mathbf{j}\|)}{2}}$$

and the amount of dependence among $X_{\mathbf{i}}$ and the variables $X_{\mathbf{j}}$, $\mathbf{j} \in V^{(n)}(\mathbf{i})$, is given by

$$\begin{aligned} h(n) &= \nu_F\left(\mathbf{i}, V^{(n)}(\mathbf{i})\right) \\ &= \frac{1}{2}-\frac{1}{8n}\left(\frac{4}{2+\sqrt{\frac{1-\exp(-n\sqrt{2})}{2}}}+\frac{4}{2+\sqrt{\frac{1-\exp(-n)}{2}}}+\sum_{i=1}^{n-1}\frac{8}{2+\sqrt{\frac{1-\exp(-\sqrt{n^2+i^2})}{2}}}\right). \end{aligned}$$

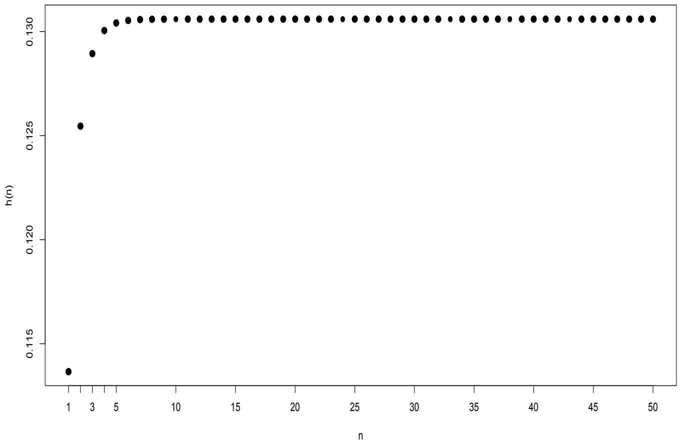

Figure 3.2. Quantifying dependence between the regions $\{\mathbf{i}\}$ and $V^{(n)}(\mathbf{i})$ in the Schlather's model

Example 3.2 (Geometric Gaussian model) Let $\mathbf{X}$ be a Geometric Gaussian random field. The bivariate distribution is given by

$$P(X_{\mathbf{i}} \leq x, X_{\mathbf{j}} \leq y) = \exp\left(-\frac{1}{x}\Phi\left(\frac{a}{2}+\frac{1}{a}\log\frac{y}{x}\right) - \frac{1}{y}\Phi\left(\frac{a}{2}+\frac{1}{a}\log\frac{x}{y}\right)\right), \quad x, y \in \mathbb{R},$$

where Φ is the standard normal distribution function and

$$a^2 = 2\sigma^2\left(1-\rho(\|\mathbf{i}-\mathbf{j}\|)\right),$$

where $\rho(.)$ is the covariance function of the Gaussian process. Let us consider $\sigma = 1$, $\mathbf{A} = \{\mathbf{i}, \mathbf{j}\}$ and $\mathbf{B} = \{\mathbf{i}+(s,0), \mathbf{j}+(s,0)\}$, $s \in \mathbb{N}$.

We have

$$\varepsilon(\mathbf{i},\mathbf{j}) = 2\Phi\left(\sqrt{\frac{1-\rho(\sqrt{(i_1-j_1)^2+(i_2-j_2)^2})}{2}}\right)$$

and

$$\begin{aligned}\nu_F(\mathbf{A},\mathbf{B}) &= \frac{1}{2}-\frac{1}{4}\left(\frac{1}{1+\Phi\left(\sqrt{\frac{1-\rho(|s|)}{2}}\right)} + \frac{1}{1+2\Phi\left(\sqrt{\frac{1-\rho(\sqrt{(-j_1+i_1+s)^2+(i_2-j_2)^2})}{2}}\right)}\right.\\ &\left.+\frac{1}{1+2\Phi\left(\sqrt{\frac{1-\rho(\sqrt{(j_1-i_1+s)^2+(i_2-j_2)^2})}{2}}\right)}\right).\end{aligned}$$

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