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A deflation method for regular matrix pencils

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ABSTRACT

A generalization of the concept of eigenvalue is introduced for a matrix pencil and it is called eigenpencil; an eigenpencil is a pencil itself and it contains part of the spectral information of the matrix pencil. A Wielandt type deflation procedure for regular matrix pencils is developed, using eigenpencils and supposing that they can have both finite and infinite eigenvalues. A numerical example illustrates the proposed method.

1. Introduction

The deflation process is linked to a large number of methods for determination of eigenvalues and the Wielandt one is probably the most classical [7]. We present here a variant of this deflation applied to regular matrix pencils.

First we recall some fundamental results about regular matrix pencils [1].

Let $\lambda B - A$, where A and B are complex matrices, be a regular matrix pencil of order n. There are matrices S and T of full rank and of dimensions $n \times r$ and $n \times t$ respectively, such that $0 \leqslant r, t$ and $r + t \leqslant n$ with

$$AS = BSJ \quad \text{and} \quad BT = ATN,$$

if and only if all eigenvalues of J are finite eigenvalues of $\lambda B - A$ and the eigenvalue zero of the nilpotent matrix N corresponds to the infinite eigenvalue of $\lambda B - A$, with BS and AT being also of full rank.

Moreover if $r + t = n$, the matrix

$$(\lambda I - J) \oplus (\lambda N - I) \tag{1}$$

is equivalent to $\lambda B - A$ and if J and N are in Jordan form, the matrix (1) is the Kronecker form of $\lambda B - A$. In this case we will refer J and N as being the finite and infinite forms of $\lambda B - A$.

For each finite eigenvalue λ_k of $\lambda B - A$, the number of respective Jordan blocks in J is the geometric multiplicity. The orders of these blocks are the partial multiplicities and the sum of these gives the algebraic multiplicity.

If J_{λ_k} is a diagonal block matrix with the all Jordan blocks of λ_k, then there exist V_{λ_k} and W_{λ_k} of full rank, such that

$$(\lambda B - A)V_{\lambda_k} = W_{\lambda_k}(\lambda I - J_{\lambda_k}).$$

Similarly, for the infinite eigenvalue we have the same definitions for the multiplicities and

$$(\lambda B - A)V_N = W_N(\lambda N - I),$$

where N is a nilpotent Jordan matrix, with V_N and W_N being of full rank.

Furthermore, if

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$$AS_1 = BS_1J_1 \quad \text{and} \quad BT_1 = AT_1N_1,$$

with J_1 being a Jordan block, N_1 being a nilpotent Jordan block and S_1 and T_1 being of full rank, then we call (S_1,J_1) and (T_1,N_1) finite and infinite eigenpairs of $\lambda B - A$ respectively.

We now mention some literature concerning our subject. The basic theory for matrix pencils, or simply pencils, is formalized in [1], scalar deflation procedures for pencils have a development in [5]. Furthermore, generalized deflation for complex matrices have been studied in [2,7,4], while equivalence relations between a pencil and a triangular form with pencils of small orders can be found in [6,3].

This article is organized as follows: in Section 2 a basic theory for eigenpencils is introduced. In Section 3 a deflation method is developed; first, separately for the finite and infinite parts, and then combined in a unique deflation. Section 4 contains a numerical example and in Section 5 the conclusions are present.

2. Eigenpencil theory

The equation considered in the next definition is not particularly new; what is new is the formulation and the possibility of manipulation that it permits.

Definition 2.1. Let $\lambda B - A$ be a matrix pencil of order n. If a matrix pencil $\lambda Y - X$ of order m, with $m < n$ is such that

$$(\lambda B - A)V = W(\lambda Y - X), \tag{2}$$

where V and W are matrices of full rank of dimension $n \times m$, we call $\lambda Y - X$ an eigenpencil of $\lambda B - A$.

Remark 2.1. If $det(\lambda Y - X) \not\equiv 0$, the eigenpencil is regular, otherwise is singular. In particular, the eigenpencil $\lambda K - 0_m$ is regular if K is a nonsingular matrix, and $0_m = (\lambda 0_m - 0_m)$ is a singular eigenpencil.

Direct consequences of Definition 2.1 are:

Theorem 2.1. *Any matrix pencil equivalent to an eigenpencil $\lambda Y - X$ of a matrix pencil $\lambda B - A$ is also an eigenpencil of $\lambda B - A$.*

Theorem 2.2. *If $\lambda Y - X$ is an eigenpencil of $\lambda B - A$, then there exist matrices L and G of full rank such that*

$$G(\lambda B - A) = (\lambda Y - X)L.$$

In addition, L and G can be chosen with $LV = I$ and $GW = I$.

Theorem 2.3. *An eigenpencil of a regular matrix pencil is also regular and a matrix pencil is singular if and only if it has a singular eigenpencil.*

We assume now that $\lambda B - A$ is regular. The next theorem establishes the strong relation between pencil and eigenpencil, under this assumption.

Theorem 2.4. *Let $\lambda B - A$ and $\lambda Y - X$ be regular matrices pencils of orders n and m respectively, with $m < n$, then $\lambda Y - X$ is an eigenpencil of $\lambda B - A$, if and only if the eigenvalues of $\lambda Y - X$ are also eigenvalues of $\lambda B - A$, and for each common finite eigenvalue λ_1, with partial multiplicities $\alpha_{e_1}, \alpha_{e_2}, \ldots, \alpha_{e_k}$ in $\lambda Y - X$ and $\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_l}$ in $\lambda B - A$ and for a common infinite eigenvalue, with partial multiplicities $\beta_{e_1}, \beta_{e_2}, \ldots, \beta_{e_t}$ in $\lambda Y - X$ and $\beta_{p_1}, \beta_{p_2}, \ldots, \beta_{p_u}$ in $\lambda B - A$, where the partial multiplicities are in decreasing order of magnitude, we have*

$$k \leqslant l \quad \text{and} \quad \alpha_{e_i} \leqslant \alpha_{p_i}; \; t \leqslant u \quad \text{and} \quad \beta_{e_j} \leqslant \beta_{p_j}.$$

Proof. Let J_p and N_p be the finite and infinite forms of $\lambda B - A$. Consider now the Kronecker form of $\lambda Y - X$, that is

$$\lambda Y - X = U \begin{bmatrix} \lambda I_r - J_e & 0 \\ 0 & \lambda N_e - I_s \end{bmatrix} R^{-1},$$

with R and U being nonsingular.

From Definition 2.1 we have $(\lambda B - A)V = W(\lambda Y - X)$, and so

$$(\lambda B - A)VR = WU \begin{bmatrix} \lambda I_r - J_e & 0 \\ 0 & \lambda N_e - I_s \end{bmatrix}$$

and VR and WU are of full rank. So it is clear that the diagonal blocks of J_e and N_e are principal submatrices of the diagonal blocks of J_p and N_p respectively, thus for a finite eigenvalue λ_1 each $J_{e_{1_i}},\ i = 1, \ldots, k$ has a correspondent block in one of the $J_{p_{1_j}},\ j = 1, \ldots, l$, hence $k \leqslant l$. Furthermore, for simplicity, supposing that the blocks are in decreasing order, we have $\alpha_{e_i} \leqslant \alpha_{p_i}$.

The same argument is valid for the infinite eigenvalue.

Conversely, we suppose that each finite eigenvalue and the infinite eigenvalue of $\lambda Y - X$, are also of $\lambda B - A$ with

$$k \leqslant l \quad \text{and} \quad \alpha_{e_i} \leqslant \alpha_{p_i};$$
$$t \leqslant u \quad \text{and} \quad \beta_{e_j} \leqslant \beta_{p_j},$$

so we can write $AV_1 = BV_1 J_e$ and $BV_2 = AV_2 N_e$, thus

$$(\lambda B - A)[V_1 V_2] = [BV_1 AV_2]\begin{bmatrix} \lambda I_r - J_e & 0 \\ 0 & \lambda N_e - I_s \end{bmatrix},$$

writing $V = [V_1 V_2]$ and $W = [BV_1 AV_2]$, it follows:

$$(\lambda B - A)V = WU^{-1}(\lambda Y - X)R$$

and hence

$$(\lambda B - A)VR^{-1} = WU^{-1}(\lambda Y - X),$$

where VR^{-1} and WU^{-1} are of full rank, then $\lambda Y - X$ is an eigenpencil of $\lambda B - A$. □

3. Deflation method

We draw attention to the fact that in this section the algebraic multiplicities of eigenvalues will be taken into account, so it means that we can say eigenvalues even when it refers to the same eigenvalue.

Now we present a block deflation procedure for regular matrix pencils. First we consider the finite and the infinite cases separately, then we join them in a single deflation formula.

- *The finite case*:
  Considering $\lambda B - A$ a regular matrix pencil and $\lambda Y - X$ being its respective eigenpencil. Then supposing that X_1 of order r is any matrix similar to the finite form of $\lambda Y - X$, that is

$$AV_1 = BV_1 X_1$$

for an $n \times r$ matrix V_1 of full rank, we define

$$\begin{cases} \overline{A} = A - BV_1 X_1 L_1, \\ \overline{B} = B \end{cases} \tag{3}$$

in which L_1 is any $r \times n$ matrix such that $L_1 V_1 = I_r$. It can be verified that the deflated pencil $\lambda \overline{B} - \overline{A}$ has all eigenvalues of $\lambda B - A$ except the finite eigenvalues of $\lambda Y - X$, which have been replaced by zeros.

- *The infinite case*:
  In a similar way we suppose that X_2 of order t is any matrix similar to the infinite form of $\lambda Y - X$ and V_2 is an $n \times t$ matrix of full rank with

$$BV_2 = AV_2 X_2.$$

We define

$$\begin{cases} \breve{A} = A - AV_2 L_2, \\ \breve{B} = B - AV_2(X_2 - I_t)L_2 \end{cases} \tag{4}$$

in which L_2 of order $t \times n$ is such that $L_2 V_2 = I_t$. Also, it can be verified that the deflated pencil $\lambda \breve{B} - \breve{A}$ has all eigenvalues of $\lambda B - A$ except the infinite eigenvalues of $\lambda Y - X$, which have been replaced by zeros.

- *The general case*:
  Putting together (3) and (4) we obtain

$$\begin{cases} \hat{A} = A - BV_1 X_1 L_1 - AV_2 L_2, \\ \hat{B} = B - AV_2(X_2 - I_t)L_2 \end{cases}$$

With these two equations, taking $W_1 = BV_1$ and $W_2 = AV_2$, which are of full rank due to the fact that $\lambda B - A$ are regular, we construct the pencil

$$\begin{aligned} \lambda \hat{B} - \hat{A} &= \lambda B - A - \lambda AV_2(X_2 - I_t)L_2 + BV_1 X_1 L_1 + AV_2 L_2 = \lambda B - A - \lambda W_2(X_2 - I_t)L_2 + W_1 X_1 L_1 + W_2 L_2 \\ &= \lambda B - A - [\, W_1 \quad W_2 \,]\begin{bmatrix} -X_1 & 0 \\ 0 & \lambda(X_2 - I_t) - I_t \end{bmatrix}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \end{aligned}$$

so, if we write $V = [\,V_1 \quad V_2\,]$, $W = [\,W_1 \quad W_2\,]$ and $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, it follows that $LV = I_m$ and we get

$$\lambda\hat{B} - \hat{A} = \lambda B - A - W\begin{bmatrix} -X_1 & 0 \\ 0 & \lambda(X_2 - I_t) - I_t \end{bmatrix}L,$$

which has all eigenvalues (finite and infinite) of $\lambda B - A$ except those of $\lambda Y - X$.

Now, considering that there exist P and Q nonsingular matrices of order m, such that

$$\lambda\hat{B} - \hat{A} = \lambda B - A - WP\begin{bmatrix} -J_e & 0 \\ 0 & \lambda(N_e - I_t) - I_t \end{bmatrix}QL,$$

where J_e and N_e are respectively the finite and infinite forms of $\lambda Y - X$, it follows:

$$\lambda\hat{B} - \hat{A} = \lambda B - A - WP\left(\begin{bmatrix} \lambda I_r - J_e & 0 \\ 0 & \lambda N_e - I_t \end{bmatrix} - \lambda I_m\right)QL = \lambda B - A - W\left(P\begin{bmatrix} \lambda I_r - J_e & 0 \\ 0 & \lambda N_e - I_t \end{bmatrix}Q - \lambda PQ\right)L.$$

Then taking $K = PQ$ and defining

$$\lambda Y' - X' = P\begin{bmatrix} \lambda I_r - J_e & 0 \\ 0 & \lambda N_e - I_t \end{bmatrix}Q,$$

which is equivalent to $\lambda Y - X$, that is, it has the same Kronecker form and considering, by Theorem 2.1, that it is also an eigenpencil of $\lambda B - A$, we can discard the prime symbol and write

$$\lambda\hat{B} - \hat{A} = \lambda B - A - W(\lambda Y - X - \lambda K)L.$$

This deflation is formalized in the next theorem in a more general context, where we do not need to assume that K is nonsingular.

Theorem 3.1. *Let $\lambda B - A$ be a regular matrix pencil and let $\lambda Y - X$ be an eigenpencil, that is $(\lambda B - A)V = W(\lambda Y - X)$, with V and W of full rank. If*

$$\lambda\hat{B} - \hat{A} = \lambda B - A - W(\lambda Y - X - \lambda K)L, \tag{5}$$

for an arbitrary $m \times m$ matrix K and L is such that $LV = I_m$, then

(i) *λK is an eigenpencil of $\lambda\hat{B} - \hat{A}$ corresponding to V and W;*
(ii) *$\lambda\hat{B} - \hat{A}$ has the eigenvalues (both finite and infinite) of $\lambda B - A$ except those (finite and infinite) of $\lambda Y - X$.*

Proof.

(i)

$$(\lambda\hat{B} - \hat{A})V = (\lambda B - A)V - W(\lambda Y - X - \lambda K)LV = W(\lambda Y - X) - W(\lambda Y - X - \lambda K) = W(\lambda Y - X - \lambda Y + X + \lambda K) = W(\lambda K).$$

(ii) The relationship between the partial multiplicities of the eigenvalues of $\lambda Y - X$ and of $\lambda B - A$ is stated in Theorem 2.4.

First we consider the case where the different eigenvalues of $\lambda Y - X$ have the same partial multiplicities in $\lambda B - A$.

Let now λ_p be a finite eigenvalue of $\lambda B - A$, which it is not an eigenvalue of $\lambda Y - X$, that is $det(\lambda_p B - A) = 0$ and $det(\lambda_p Y - X) \neq 0$. We suppose now $\lambda_\epsilon = \lambda_p + \epsilon, \epsilon > 0$, such that λ_ϵ is not eigenvalue of $\lambda B - A$ nor of $\lambda Y - X$. Thus, from

$$(\lambda_\epsilon B - A)V = W(\lambda_\epsilon Y - X)$$

it follows:

$$V(\lambda_\epsilon Y - X)^{-1} = (\lambda_\epsilon B - A)^{-1}W$$

and hence

$$\begin{aligned} det(\lambda_\epsilon\hat{B} - \hat{A}) &= det(\lambda_\epsilon B - A - W(\lambda_\epsilon Y - X - \lambda_\epsilon K)L) = det(\lambda_\epsilon B - A)det[I_n - (\lambda_\epsilon B - A)^{-1}W(\lambda_\epsilon Y - X - \lambda_\epsilon K)L] \\ &= det(\lambda_\epsilon B - A)det[I_n - V(\lambda_\epsilon Y - X)^{-1}(\lambda_\epsilon Y - X - \lambda_\epsilon K)L] = det(\lambda_\epsilon B - A)det[I_n - V(I_m - (\lambda_\epsilon Y - X)^{-1}\lambda_\epsilon K)L] \\ &= det(\lambda_\epsilon B - A)det[I_m - LV(I_m - (\lambda_\epsilon Y - X)^{-1}\lambda_\epsilon K)] = det(\lambda_\epsilon B - A)det[I_m - (I_m - (\lambda_\epsilon Y - X)^{-1}\lambda_\epsilon K)] \\ &= det(\lambda_\epsilon B - A)det[(\lambda_\epsilon Y - X)^{-1}\lambda_\epsilon K] = det(\lambda_\epsilon B - A)det(\lambda_\epsilon Y - X)^{-1}det(\lambda_\epsilon K) \\ &= det(\lambda_\epsilon B - A)det(\lambda_\epsilon Y - X)^{-1}(\lambda_\epsilon)^m det(K) \end{aligned}$$

so we have from $\lim_{\epsilon\to 0}\lambda_\epsilon = \lambda_p$, that $\lim_{\epsilon\to 0} det(\lambda_\epsilon B - A) = 0$ and then $\lim_{\epsilon\to 0} det(\lambda_\epsilon \hat{B} - \hat{A}) = 0$.

For the infinite eigenvalue we consider the pencil $\mu A - B$ and μ_t being the respective zero eigenvalue and not being an eigenvalue of $\mu X - Y$ and thus we make a similar development as above.

We consider now the case where there are some eigenvalues in $\lambda Y - X$ in which the partial multiplicities are greater in $\lambda B - A$ (this means that these eigenvalues, finite or infinite, will remain in the deflated pencil with smaller multiplicities).

Thus we suppose that there are V^s and W^s of dimensions $n \times s$, s = n−m, such that

$$(\lambda B - A)\begin{bmatrix} V & V^s \end{bmatrix} = \begin{bmatrix} W & W^s \end{bmatrix}\begin{bmatrix} \lambda Y - X & \lambda Y^{s_1} - X^{s_1} \\ 0 & \lambda Y^{s_2} - X^{s_2} \end{bmatrix} = \begin{bmatrix} W & W^s \end{bmatrix}(\lambda Y^{(a)} - X^{(a)}),$$

in which $\begin{bmatrix} V & V^s \end{bmatrix}$ and $\begin{bmatrix} W & W^s \end{bmatrix}$ are nonsingular matrices and the pencil $\lambda Y^{(a)} - X^{(a)}$ is equivalent to $\lambda B - A$.

Now, we define $\hat{V}^s = V^s - VLV^s$, and using Theorem 2.2, it follows:

$$\begin{aligned}(\lambda\hat{B} - \hat{A})\hat{V}^s &= (\lambda\hat{B} - \hat{A})V^s - (\lambda\hat{B} - \hat{A})VLV^s = (\lambda B - A)V^s - W(\lambda Y - X)LV^s + W(\lambda K)LV^s - W(\lambda K)LV^s \\ &= W(\lambda Y^{s_1} - X^{s_1}) + W^s(\lambda Y^{s_2} - X^{s_2}) - WG(\lambda B - A)V^s \\ &= W(\lambda Y^{s_1} - X^{s_1}) + W^s(\lambda Y^{s_2} - X^{s_2}) - WGW(\lambda Y^{s_1} - X^{s_1}) - WGW^s(\lambda Y^{s_2} - X^{s_2}) \\ &= W^s(\lambda Y^{s_2} - X^{s_2}) - WGW^s(\lambda Y^{s_2} - X^{s_2}) = (W^s - WGW^s)(\lambda Y^{s_2} - X^{s_2}),\end{aligned}$$

with $\hat{W}^s = W^s - WGW^s$, and considering that $\begin{bmatrix} V & V^s \end{bmatrix}$ and $\begin{bmatrix} W & W^s \end{bmatrix}$ are of full rank, it follows that $\hat{V}^s$ and $\hat{W}^s$ are also of full rank. □

Furthermore, by Theorems 2.4 and 3.1 the following can be verified:

Corollary 3.1. *If K is nonsingular,*

$$(S_{p_1}, J_{p_1}), (S_{p_2}, J_{p_2}), \ldots, (S_{p_r}, J_{p_r}) \quad \text{and} \quad (T_{p_1}, N_{p_1}), (T_{p_2}, N_{p_2}), \ldots, (T_{p_q}, N_{p_q})$$

are the finite and infinite eigenpairs of $\lambda B - A$ *and*

$$(S_{e_1}, J_{e_1}), (S_{e_2}, J_{e_2}), \ldots, (S_{e_g}, J_{e_g}) \quad \text{and} \quad (T_{e_1}, N_{e_1}), (T_{e_2}, N_{e_2}), \ldots, (T_{e_h}, N_{e_h})$$

are the finite and the infinite eigenpairs of $\lambda Y - X$*, all in an arbitrary order, then for a suitable order the eigenpairs of* $\lambda\hat{B} - \hat{A}$ *will be:*

I-$(S'_{p_1}, 0_{k_1}), (S'_{p_2}, 0_{k_2}), \ldots, (S'_{p_g}, 0_{k_g})$,

where $S'_{p_i},\ i = 1, \ldots, g$, *is formed by the first* k_i *columns of* S_{p_i} *with* k_i *being the order of* J_{e_i}.

II-$(\hat{S}_{p_1}, \hat{J}_{p_1}), (\hat{S}_{p_2}, \hat{J}_{p_2}), \ldots, (\hat{S}_{p_g}, \hat{J}_{p_g})$,

where the Jordan block $\hat{J}_{p_i},\ i = 1, \ldots, g$ *of order* $l_i - k_i$ *is a principal submatrix of* J_{p_i} *of order* l_i.

III-$(\hat{S}_{p_{g+1}}, J_{p_{g+1}}), (\hat{S}_{p_{g+2}}, J_{p_{g+2}}), \ldots, (\hat{S}_{p_r}, J_{p_r})$,

where $J_{p_i},\ i = g + 1, \ldots, r$ *is a Jordan block of the original pencil.*

IV-$(T'_{p_1}, 0_{s_1}), (T'_{p_2}, 0_{s_2}), \ldots, (T'_{p_h}, 0_{s_h})$,

where $T'_{p_i},\ i = 1, \ldots, h$, *is formed by the first* s_i *columns of* T_{p_i} *with* s_i *being the order of* N_{p_i}.

V-$(\hat{T}_{p1}, \hat{N}_{p_1}), (\hat{T}_{p2}, \hat{N}_{p_2}), \ldots, (\hat{T}_{p_h}, \hat{N}_{p_h})$,

where the nilpotent Jordan block $\hat{N}_{p_i},\ i = 1, \ldots, h$, *is of order* $t_i - s_i$, *with* t_i *being the order of* N_{p_i}.

VI-$(\hat{T}_{p_{h+1}}, N_{p_{h+1}}), (\hat{T}_{p_{h+2}}, N_{p_{h+2}}), \ldots, (\hat{T}_{p_q}, N_{p_q})$,

where $N_{p_i},\ i = h + 1, \ldots, q$, *is a nilpotent Jordan block of the original pencil.*

We note that in Theorem 3.1, if we consider K nonsingular, the eigenvalues of λK are all zeros and it means that the deflated pencil $\lambda\hat{B} - \hat{A}$ is regular. Furthermore, the eigenpairs of items I and IV in Corollary 3.1 are finite.

On the other hand, if $\lambda\hat{B} - \hat{A}$ is defined with K singular, $\lambda\hat{B} - \hat{A}$ will be a singular pencil and the structures of items I and IV in Corollary 3.1 will be singular. We do not pursue a definition for such structures here. However, we will see below that defining $K = 0_m$ can be useful for continuing the deflation to the next step.

Reporting again to Theorem 3.1, V and W are of full rank, so there are m rows which are linearly independent and for the sake of simplicity, we can suppose that these m rows are the first rows. Hence, with these rows we can construct nonsingular matrices H_1 and H_2 of order m, such that VH_1^{-1} and WH_2^{-1} are normalized with the first m rows being the block I_m.

Thus we can enunciate.

Corollary 3.2. *If $K = 0_m, V_{(m)}H_1^{-1} = W_{(m)}H_2^{-1} = I_m$ and $H_2(\lambda Y - X)H_1^{-1}L = (\lambda B - A)_{(m)}$, where the subscripts (m) indicate the first m rows,*

$$\lambda\hat{B} - \hat{A} = \lambda B - A - WH_2^{-1}(\lambda B - A)_{(m)}$$

in which

(i) *the first m rows are null;*
(ii) *we can construct a pencil $\lambda\tilde{B} - \tilde{A}$ of order $n - m$ from $\lambda\hat{B} - \hat{A}$ by taking off the first m rows and the first m columns, such that $\lambda\tilde{B} - \tilde{A}$ has only the eigenvalues of $\lambda B - A$ that are not eigenvalues of $\lambda Y - X$.*

Proof.

(i) Let

$$(\lambda B - A)VH_1^{-1} = WH_2^{-1}H_2(\lambda Y - X)H_1^{-1},$$

where $H_2(\lambda Y - X)H_1^{-1}$ is equivalent to $\lambda Y - X$, and so it is also an eigenpencil of $\lambda B - A$.
Now, we write $V' = VH_1^{-1},\ W' = WH_2^{-1}$ and $\lambda Y' - X' = H_2(\lambda Y - X)H_1^{-1}$.
We verify that

$$LV' = (\lambda Y' - X')^{-1}(\lambda B - A)_{(m)}V' = (\lambda Y' - X')^{-1}(W')_{(m)}(\lambda Y' - X') = (\lambda Y' - X')^{-1}(\lambda Y' - X') = I_m.$$

Thus considering $K = 0$ in Theorem 3.1,

$$\lambda\hat{B} - \hat{A} = \lambda B - A - W'(\lambda Y' - X')L = \lambda B - A - W'(\lambda B - A)_{(m)},$$

then the first m rows of $\lambda\hat{B} - \hat{A}$ are

$$(\lambda\hat{B} - \hat{A})_{(m)} = (\lambda B - A)_{(m)} - (W')_{(m)}(\lambda B - A)_{(m)} = 0_{m\times n}.$$

(ii) By Theorem 3.1, $\lambda\hat{B} - \hat{A}$ will have $n - m$ eigenvalues from $\lambda B - A$ that are not eigenvalues of $\lambda Y' - X'$, and thus, they are not eigenvalues of $\lambda Y - X$ either.
Therefore there are matrices V_1 and W_1 of full rank such that

$$(\lambda B - A)V_1 = W_1(\lambda Y_1 - X_1),$$

where $\lambda Y_1 - X_1$ has the remaining $n - m$ eigenvalues.
Let

$$\hat{W} = W_1 - W'(W_1)_{(m)} \quad \text{and} \quad \hat{V} = V_1 - V'(V_1)_{(m)},$$

then

$$(\hat{W})_{(m)} = 0_m \quad \text{and} \quad (\hat{V})_{(m)} = 0_m$$

and so

$$\begin{aligned}(\lambda\hat{B} - \hat{A})\hat{V} &= (\lambda B - A)\hat{V} - W'(\lambda B - A)_{(m)}\hat{V} = (\lambda B - A)(V_1 - V'(V_1)_{(m)}) - W'(\lambda B - A)_{(m)}(V_1 - V'(V_1)_{(m)})\\ &= (\lambda B - A)V_1 - (\lambda B - A)V'(V_1)_{(m)} - W'(\lambda B - A)_{(m)}V_1 + W'(\lambda B - A)_{(m)}V'(V_1)_{(m)}\\ &= W_1(\lambda Y_1 - X_1) - W'(\lambda Y' - X')(V_1)_{(m)} - W'(W_1)_{(m)}(\lambda Y_1 - X_1) + W'(W')_{(m)}(\lambda Y' - X')(V_1)_{(m)}\\ &= W_1(\lambda Y_1 - X_1) - W'(W_1)_{(m)}(\lambda Y_1 - X_1) = (W_1 - W'(W_1)_{(m)})(\lambda Y_1 - X_1) = \hat{W}(\lambda Y_1 - X_1).\end{aligned}$$

Hence

$$(\lambda\hat{B} - \hat{A})\hat{V} = \hat{W}(\lambda Y_1 - X_1),$$

that is

$$\begin{bmatrix} 0_m & 0_{(n-m)\times m} \\ * & \lambda\tilde{B} - \tilde{A} \end{bmatrix}\begin{bmatrix} 0_m \\ \tilde{V} \end{bmatrix} = \begin{bmatrix} 0_m \\ \tilde{W} \end{bmatrix}(\lambda Y_1 - X_1)$$

and thus,

$$(\lambda\tilde{B} - \tilde{A})\tilde{V} = \tilde{W}(\lambda Y_1 - X_1). \quad \square$$

Besides that, $\lambda\tilde{B}-\tilde{A}$ will have the Kronecker form according to II, III, V, and VI of Corollary 3.1.

Concluding, as we mentioned before, now we have a deflated pencil $\lambda\hat{B}-\hat{A}$ singular, but $\lambda\tilde{B}-\tilde{A}$ is a regular pencil, which permits us to continue the deflation process.

4. Numerical example

If we consider a regular matrix pencil containing finite and infinite forms, then we can deflate it with 3 possibilities, where an eigenpencil can have only finite eigenvalues; only infinite ones; or both, finite and infinite. In the next example it is explored the case where the eigenpencil has both finite and infinite eigenvalues.

The normalization is carried out with the first m rows which we can construct a nonsingular block simultaneously in V and W. If it is not possible to find it, we can always multiply V (or W) for a nonsingular elementary matrix to interchange some of the rows.

Let

$$\lambda B-A=\begin{bmatrix} 3\lambda-1 & 2-3\lambda & \lambda-2 & 2\lambda-1 & 1-2\lambda & \lambda-3 & 3\lambda-4 & 1\\ -\lambda & 3\lambda-1 & 0 & 1-\lambda & \lambda & 2-2\lambda & 1-2\lambda & \lambda\\ 11\lambda-18 & 14-5\lambda & \lambda-2 & 6\lambda-10 & 12-6\lambda & 4-4\lambda & 2\lambda-5 & 3\\ 3-2\lambda & -\lambda-2 & 0 & 2-\lambda & -1 & 3\lambda-1 & \lambda+1 & 1-2\lambda\\ 10-5\lambda & 5\lambda-7 & 2-\lambda & 5-2\lambda & 4\lambda-7 & -\lambda & 4-3\lambda & 2\lambda-1\\ \lambda-3 & -2\lambda & 0 & -1 & 1-\lambda & 2\lambda-1 & 2\lambda-2 & -\lambda-1\\ \lambda-1 & 3-3\lambda & \lambda-2 & \lambda-1 & 3-2\lambda & 3\lambda-4 & 3\lambda-4 & 3-2\lambda\\ 3\lambda-11 & 5-\lambda & 0 & \lambda-4 & 6-2\lambda & \lambda+1 & 2\lambda-5 & 1-\lambda \end{bmatrix}$$

whose Kronecker canonical form is

$$\begin{bmatrix} -1 & \lambda & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda-1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \lambda-1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \lambda-2 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda-2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-2 \end{bmatrix}.$$

The pencil

$$\lambda Y_1-X_1=\begin{bmatrix} 3\lambda-8 & 2-2\lambda\\ 3\lambda-2 & 8-2\lambda \end{bmatrix}$$

is an eigenpencil of $\lambda B-A$. We have that

$$V_1=\begin{bmatrix} -2 & -2\\ 0 & 0\\ -5 & 0\\ 0 & 0\\ -2 & -2\\ -2 & -2\\ 2 & 2\\ 0 & 0 \end{bmatrix}\quad\text{and}\quad W_1=\begin{bmatrix} -1 & 0\\ \frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & -\frac{2}{3}\\ 0 & 0\\ 1 & 0\\ -\frac{1}{3} & \frac{1}{3}\\ -\frac{2}{3} & -\frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

are of full rank and $(\lambda B-A)V_1=W_1(\lambda Y_1-X_1)$.

The eigenpencil λY_1-X_1 has one infinite eigenvalue and the finite eigenvalue 2. Its Kronecker canonical form is $\begin{bmatrix} -1 & 0\\ 0 & \lambda-2 \end{bmatrix}$.

First we normalize V_1 and W_1, using the first and the third rows, so we have

$$V_1' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad W_1' = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Thus we consider the first and the third rows of $\lambda B - A$,

$$(\lambda B - A)_{(1,3)} = \begin{bmatrix} 3\lambda - 1 & 2 - 3\lambda & \lambda - 2 & 2\lambda - 1 & 1 - 2\lambda & \lambda - 3 & 3\lambda - 4 & 1 \\ 11\lambda - 18 & 14 - 5\lambda & \lambda - 2 & 6\lambda - 10 & 12 - 6\lambda & 4 - 4\lambda & 2\lambda - 5 & 3 \end{bmatrix}$$

so,

$$\lambda\hat{B_1} - \hat{A_1} = \lambda B - A - W_1'(\lambda B - A)_{(1,3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{17}{2} - 5\lambda & 4\lambda - 7 & 0 & \frac{11}{2} - 3\lambda & 3\lambda - \frac{11}{2} & \frac{\lambda-3}{2} & -\frac{3}{2}(\lambda - 1) & \lambda - 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 - 2\lambda & -\lambda - 2 & 0 & 2 - \lambda & -1 & 3\lambda - 1 & \lambda + 1 & 1 - 2\lambda \\ 9 - 2\lambda & 2\lambda - 5 & 0 & 4 & 2(\lambda - 3) & -3 & 0 & 2\lambda \\ 5\lambda - \frac{23}{2} & 6 - 3\lambda & 0 & 2\lambda - \frac{11}{2} & \frac{13}{2} - 3\lambda & \frac{5-\lambda}{2} & \frac{1}{2}(3\lambda - 5) & -\lambda \\ \frac{17}{2} - 6\lambda & \lambda - 5 & 0 & \frac{9}{2} - 3\lambda & 2\lambda - \frac{7}{2} & \frac{9(\lambda-1)}{2} & \frac{\lambda+1}{2} & 1 - 2\lambda \\ -\lambda - \frac{5}{2} & -1 & 0 & \frac{1}{2} - \lambda & \frac{1}{2} & \frac{1}{2}(7\lambda - 5) & \frac{1}{2}(5\lambda - 9) & -\lambda \end{bmatrix}.$$

We take off the first and the third rows and columns, then we get the regular pencil

$$\lambda\tilde{B} - \tilde{A} = \begin{bmatrix} 4\lambda - 7 & \frac{11}{2} - 3\lambda & 3\lambda - \frac{11}{2} & \frac{\lambda-3}{2} & -\frac{3}{2}(\lambda - 1) & \lambda - 1 \\ -\lambda - 2 & 2 - \lambda & -1 & 3\lambda - 1 & \lambda + 1 & 1 - 2\lambda \\ 2\lambda - 5 & 4 & 2(\lambda - 3) & -3 & 0 & 2\lambda \\ 6 - 3\lambda & 2\lambda - \frac{11}{2} & \frac{13}{2} - 3\lambda & \frac{5-\lambda}{2} & \frac{1}{2}(3\lambda - 5) & -\lambda \\ \lambda - 5 & \frac{9}{2} - 3\lambda & 2\lambda - \frac{7}{2} & \frac{9(\lambda-1)}{2} & \frac{\lambda+1}{2} & 1 - 2\lambda \\ -1 & \frac{1}{2} - \lambda & \frac{1}{2} & \frac{1}{2}(7\lambda - 5) & \frac{1}{2}(5\lambda - 9) & -\lambda \end{bmatrix},$$

whose Kronecker canonical form is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda - 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & \lambda - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 2 \end{bmatrix}.$$

5. Conclusions

We used this new definition: eigenpencil, to extend the Wielandt deflation for regular matrix pencils to a block context and to permit the deflation of finite and infinite eigenvalues at the same time. We supposed that an eigenpencil is known without giving a method to compute them, but we believe that in future works this can be achieved with the generalization of some scalar methods.

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