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A Model Structured in Classes for the Study of Student Population Evolution

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Abstract: This paper presents an application of a metapopulation model structured in classes for the study of population evolution of a network of schools. The global dynamics are considered in two parts, the local, where the evolution for each grade in a cycle is accounted, and the metapopulation or the movement of students between schools. Simulation of the model is presented.

Key words: metapopulation model; class structured; population evolution
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Classical population models such as the logistic growth or the Lotka-Volterra predator-prey system[1] ignore population structure by treating all individuals as equal. In any natural population, the existence of demographically fundamental differences among individuals is clear. For example, the age or the size of an individual can be crucial for its survival or in its ability to find partners. Moreover, age, size or any other physiological measurements can be responsible for triggering important processes such as reproduction[2].

In an educational institution, the dropout and completion rates depend on the student entrance year ("age"). Population biologists, demographers and ecologists use accurate models to provide predictions of population densities and how they distribute among various levels of stratification[3-4]. Governmental organizations and businesses need a constant use of reliable stage structured, or compartmental, models to project Social Security and education outlays, immigration policy, tax recipients, etc[5-6].

The Leslie matrix model is a discrete time dynamical system, named after Leslie and his pioneering work[7] designed to describe the growth of a population through its age distribution. The population is divided into n age classes of same duration and the whole population at time t is stored in an n-dimensional age distribution vector $X(t)$. Evolution of the age classes is given by $X(t+1) = LX(t)$, where L is an $n \times n$ Leslie matrix, a nonnegative matrix with nonzero entries only in the subdiagonal describing the transition from one age class to another (the ageing process) and in the first line describing the recruitment process into the first age class (birth process).

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Other distributions based on other physiological variables such as size or stage in the life cycle are considered in Ref. [2]. Similar ideas can be used to model the time evolution of a population of students in an educational institution, where the population is divided into “age” classes based on the entrance year. The “ageing” process (advancing from one level to the next one) is based on the student academic success. Clearly, in this case there is no birth of new individuals. Therefore, an external recruitment term representing the admittance of new students has to be considered such as in Ref. [8].

In the real world we rarely observe an isolated population, meaning a population closed to dispersal movements. Natural populations more often are composed of habitat patches (subpopulations) connected by migration (exchange of individuals between connected sites)[9]. The ensemble is sometimes called metapopulation. Such a network of coupled populations naturally arises in human demography because provinces, states or countries can be viewed as collections of cities connected by migratory movements. Of course these migratory movements are age class dependent as in Refs. [10-11]. For example, migration of students from one school to another depending on the year of admittance was considered in Ref. [8].

A mathematical formalism to deal with these spatially extended systems was introduced in Ref. [12]. The so-called Coupled Map Lattices (CML) are suitable for computational studies because of the discrete nature of time and space. In CML the dynamical units are located at discrete points in space, time is discrete, while the state variables are continuous. Each of these dynamical units is coupled to its neighbors forming a network. The selection of the neighbors is determined by the network topology. In many theoretical studies, regular networks (each site is equally connected to all sites) and two-nearest neighbors (linear arrays where each site is connected only with the two nearest sites) are preferred. But, in real world situations the network architecture can be very complex as in metabolic networks[13], internet networks[14] and citation network of scientists[15]. It is essential to have a characterization of the network anatomy because modifications in the network structure can cause different functional responses. For instance, the topology of social networks affects the spread of information and diseases[16].

In mathematical terms, the time evolution of network of n equal sites is given by the equations

$$x^i_{(t+1)} = (1-m)f(x^i_t) + m\sum_{i=1}^{n} c_{ij} f(x^j_t), \quad i = 1, 2, \cdots, n,$$

where x^i_t is the state of site i at time t, $f : \mathbf{R} \to \mathbf{R}$ is a smooth function modeling the local dynamics (the dynamics of an isolated site, which corresponds to $m = 0$). $m \in [0, 1]$ is the coupling parameter, and the $n \times n$, nonnegative matrix C, called connectivity matrix, with entries $c_{ij} \in [0, 1]$, $c_{ij} = 0$, $i, j = 1, 2, \cdots, n$, characterizing the network topology. In population dynamics studies, typically, the state is the population density, m is the fraction of individuals that leave a given site (called migration fraction), and c_{ij} is probability of a migrant that left site j to settle in site i. These systems can display a wide variety of dynamical phenomena ranging from simple periodic dynamics to very complicated patterns of spatiotemporal chaos[17] and even self-organization phenomena such as synchronization[18-19].

If the individuals are not equal, that is, if an age structure is considered, for instance, if the population is distributed over s age classes, then the vector

$$X^i(t) = [x_1^i(t), x_2^i(t), \cdots, x_s^i(t)]$$

provides the age distribution population vector of site i at time t, where each component $x_l^i(t)$, $l = 1, 2, \cdots, k$ is the population of age class l of site i at time t. Let $F : \mathbf{R}^k \to \mathbf{R}^k$ be a smooth function describing the local dynamics (survival, ageing and reproduction), and let $M = \mathrm{diag}(m_1, m_2, \cdots, m_k)$ be a diagonal $k \times k$ matrix, where each entry m_l, $l = 1, 2, \ldots, k$ represents the migration fraction of the age class l. The time evolution of this network of coupled populations is given by

$$X^i(t+1) = (I - M^i)F(X^i(t)) + \sum_{j=1}^{n} c_{ij} M^j F(X^j(t)),$$

where I is the $k \times k$ identity matrix. A more general case was considered in Ref. [11], relaxing some hypotheses such as having identical sites, allowing the migration fraction matrix and the local dynamics functional form to depend on the location. Moreover, they considered a rather complex network allowing the coefficients c_{ij} of the connectivity matrix to be age dependent, thereby considering an array of k networks connecting the n sites in different ways depending on the age class.

In this paper, we use the formulism described above along with the ideas proposed in Refs. [8,11,20] to model the dynamics of a network of educational institutions connected by the transfer of students between schools. Our main goal is to describe both the student success in a school with different levels of study, and the cycles each of which comprising a distinct number of grades or cycle years as well as the migration process or the movement of students among a network of schools. In Section 1, we present the mathematical model containing the local and metapopulation dynamics. In Section 2, we perform a simulation with a three-cycled structured educational system in a three schools network.

1 The model

The model is composed of two parts, the local dynamics and the metapopulation dynamics. The local dynamics has two stages: the first represents evolution of the students in a study cycle and the second represents transition for the next cycle.

The number of students in an educational institution, or simply a school, is denoted by a vector of dimension $k \times 1$,

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_k(t) \end{bmatrix},$$

where each element $X_c(t)$ is the student population vector in the study cycles $c = 1, 2, \cdots, k$.

The number of cycle years or grades in each cycle, and the size of the vectors $X_c(t)$, are in general different for each cycle.

1.1 Local dynamics

The local dynamics is represented in two stages. In the first stage we divide the student population of a s-year cycle c in approved and flunked. For this, we consider the vectors

$$A_c(t) = [a_1(t), a_2(t), \cdots, a_s(t)]^{\mathrm{T}},$$

$$R_c(t) = [r_1(t), r_2(t), \cdots, r_s(t)]^{\mathrm{T}}$$

in which the components $a_1(t), a_2(t), \cdots, a_s(t)$ of $A_c(t)$ are numbers of approved in the cycle years $1, 2, \cdots, s$ in the end of the academic year t and the components $r_1(t), r_2(t), \cdots, r_s(t)$ of $R_c(t)$ are numbers of flunked in the cycle years $1, 2, \cdots, s$ in the end of the academic year t. We note that the students which are approved in the last year of a cycle are those who finish the cycle, so they are considered in the second stage. If it is the last year of the last cycle the student leaves the school.

The number of students in the academic year $t+1$ which were approved from the year t is

$$N_c A_c(t) = [0, a_1(t), a_2(t), \cdots, a_{s-1}(t)]^{\mathrm{T}},$$

where

$$N_c = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

Thus the number of students in the cycle years $1, 2, \cdots, s$ in the academic year $t+1$ is represented by the vector

$$Y_c(t+1) = [y_1(t+1), y_2(t+1), \cdots, y_s(t+1)],$$

where the components $y_i(t+1)$, $i = 1, 2, \cdots, s$, are the sum of the number of flunked with the number of approved in the previous cycle year.

This dynamics can be expressed by the matrix equation

$$Y_c(t+1) = R_c(t) + N_c A_c(t). \tag{1}$$

Instead of the vectors $A_c(t)$ and $R_c(t)$, it is desirable to have the vector $Y_c(t+1)$ in terms of the student population $X_c(t)$ of the academic year t. This can be achieved considering that $X_c(t) = R_c(t) + A_c(t)$, so

$$Y_c(t+1) = X_c(t) - A_c(t) + N_c A_c(t). \tag{2}$$

Furthermore, let

$$t_1 = \frac{a_1(t)}{x_1(t)}, \quad t_2 = \frac{a_2(t)}{x_2(t)}, \ \cdots, \quad t_s = \frac{a_s(t)}{x_s(t)}$$

be the ratio, in the academic year t, between the approved students and the total students in each cycle year, hence we put this data in a diagonal matrix

$$T_c = \mathrm{diag}(t_1, t_2, \cdots, t_s).$$

Thus, the vector $A_c(t)$ can be written by

$$A_c(t) = T_c X_c(t). \tag{3}$$

It follows that

$$N_c A_c(t) = N_c T_c X_c(t), \tag{4}$$

now substituting the expressions (4) and (3) in (2), the first stage of the local dynamics in the academic year $t+1$ can be described by

$$Y_c(t+1) = (I_s - T_c + N_c T_c) X_c(t), \tag{5}$$

where I_s is an identity matrix of order s.

The second stage describes as follows:

(1) Recruiting in the first year of each cycle and moving of students from a cycle to the next, that is, the students who complete the last year in one cycle and enter the next cycle.

(2) Students who enter or leave in an intermediate year without coming or going from or to another school of the network, for example, foreign students, students who return to the studies, dropout students.

In terms of mathematical dynamics the second stage, the “recruiting”, could be included in the first stage, but our purpose is to have an extra tool for the simulation where we can get a more complete analysis with this separated information.

Let

$$G_c(t) = [g_1(t), g_2(t), \cdots, g_s(t)]$$

be the vector of students in the cycle year $1, 2, \cdots, s$, who leave the school in the academic year t and let

$$B_c(t+1) = [b_1(t+1), b_2(t+1), \cdots, b_s(t+1)]$$

be the vector of students in the cycle year $1, 2, \cdots, s$, who enter the school in the academic year $t+1$. Thus the second stage of the local dynamics is given by the vector

$$W_c(t+1) = B_c(t+1) - G_c(t). \tag{6}$$

Once more it is desirable to have also the vector $W_c(t+1)$ represented in terms of the vector $X_c(t)$, for this we consider the ratios

$$e_1 = \frac{b_1(t+1)}{x_1(t)}, \quad e_2 = \frac{b_2(t+1)}{x_2(t)}, \cdots, \quad e_s = \frac{b_s(t+1)}{x_s(t)},$$

and

$$d_1 = \frac{g_1(t)}{x_1(t)}, \quad d_2 = \frac{g_2(t)}{x_2(t)}, \cdots, \quad d_s = \frac{g_s(t)}{x_s(t)}.$$

Thus, we can construct the diagonal matrices

$$E_c = \mathrm{diag}(e_1, e_2, \cdots, e_s)$$

and

$$D_c = \mathrm{diag}(d_1, d_2, \cdots, d_s).$$

Then we have respectively

$$B_c(t+1) = E_c X_c(t)$$

and

$$G_c(t) = D_c X_c(t).$$

Hence, the equation (6) can be written by

$$W_c(t+1) = (E_c - D_c)X_c(t). \tag{7}$$

We sum $Y_c(t+1)$ and $W_c(t+1)$ and obtain the local dynamics for the cycle c,

$$Z_c(t+1) = Y_c(t+1) + W_c(t+1). \tag{8}$$

Now substituting equations (5) and (7) in (8), we have the student population for the cycle c in the academic year $t+1$ in terms of the student population vector $X_c(t)$:

$$Z_c(t+1) = (I_s - T_c + N_c T_c)X_c(t) + (E_c - D_c)X_c(t),$$

or equivalently

$$Z_c(t+1) = (I_s - T_c + N_c T_c + E_c - D_c)X_c(t).$$

Now considering not only the cycle c, but all the cycles $1, 2, \cdots, k$ in a school, the local dynamics can be represented by the vector

$$Z(t+1) = \begin{bmatrix} Z_1(t) \\ Z_2(t) \\ \vdots \\ Z_k(t) \end{bmatrix} = (I - T + NT + E - D)X(t),$$

where the matrices $T = \mathrm{diag}(T_c)$, $N = \mathrm{diag}(N_c)$, $E = \mathrm{diag}(E_c)$, $D = \mathrm{diag}(D_c)$ and $X = \mathrm{diag}(X_c)$, for $c = 1, 2, \cdots, k$, and I is the identity matrix of the appropriated size.

1.2 Metapopulation dynamics

We now describe the process of movement of students between schools, in general this change of school occurs in the beginning and in the end of each cycle, but some transfers can occur in the intermediate cycle years.

A typical metapopulation dynamics accounts for each site in a network, the in and out migration of the population. In our model, this is represented by means of two diagonal matrices, in which the diagonal elements will have the rate for those movements for each cycle year.

The first matrix is called the migration matrix

$$M^i = \mathrm{diag}(m_l^i),$$

where the diagonal elements m_l^i represent the rate of students in the cycle years

$$l = 1, 2, \cdots, s_1, 1, 2, \cdots, s_2, \cdots, 1, 2, \cdots, s_k$$

of the cycles $1, 2, \cdots, k$, who transfer from school i to another school without finishing the respective cycle.

The second is the interaction matrix

$$C^{ij} = \mathrm{diag}(c_l^{ij}),$$

where the diagonal elements c_l^{ij} represent the rate of students in the cycle years

$$l = 1, 2, \cdots, s_1, 1, 2, \cdots, s_2, \cdots, 1, 2, \cdots, s_k$$

of the cycles $1, 2, \cdots, k$, who transfer to school i coming from school $j \neq i$, then it is obvious that $C^{ii} = 0$.

The metapopulation process is considered after the local dynamics, this means that if a student transfer occurs in the middle of an academic year, first he is accounted as a flunk student and then considered as a migrant student, while the dropout student is considered in the second stage of the local dynamics. Furthermore, a student who graduates in a cycle of studies is not a migrant and if he enters the next cycle he will be considered in the second stage of the local dynamics.

The metapopulation in each school in the academic year $t+1$ is represented by the difference between the students transferring to school i and the students who transfer from school i to a school $j(\neq i)$,

$$\Big(\sum_{j=1}^{k} C^{ij} M^j Z^j(t+1) - M^i Z^i(t+1)\Big).$$

The global dynamics in school i in the academic year $t+1$ is the sum of local dynamics with the metapopulation dynamics

$$Z^i(t+1) + \sum_{j=1}^{k} C^{ij} M^j Z^j(t+1) - M^i Z^i(t+1).$$

Thus, our model can be represented by the population vector

$$X^i(t+1) = (I - M^i) Z^i(t+1) + \sum_{j=1}^{k} C^{ij} M^j Z^j(t+1),$$

where $Z^i(t+1)$ is the student population in school i, and $Z^j(t+1)$ is the student population in each of the remaining schools after the local dynamics.

2 Simulation

The great advantage of having the dynamics in two steps is that we can get the results of evolution in a cycle and in a school regardless of the other cycles in the same or in another school. Furthermore, the network can be assembled where a school can represent a group of schools or even the school population of a geographical region, permitting a global view of a large scale student population.

The data used in our simulation were adopted from the official data of Education Ministry of Portugal. This data can be used to perform simulations for the coming years, where different scenarios can be built including better and worst projections.

We assume in our simulation a network of three schools with three cycles of studies, cycles 1, 2 and 3, but with the third school having only cycles 1 and 2. The classification presented is a typical 9-year (grades) fundamental studies in Europe with the three cycles having 4, 2 and 3 years respectively.

School 1 is larger than School 2, and School 2 is larger than School 3. In terms of student success, the order is reverse, thus School 3 is the best, then comes School 2, and School 1 has the worst rate. This order is also valid for the school dropout, the smaller the school, the less is the school dropout. Recruiting in the first year has almost the same rate in School 1 and School 2, and it is a little bit higher in School 3.

Migration among the schools, the metapopulation, is greater in the direction to the best school, meaning that more students move from School 1, and more students move to School 3.

We can observe the behavior described above in the next figures, Fig. 1 has the local dynamics for cycle 1 and Fig. 2 has the respective metapopulation dynamics. Fig. 3 and Fig. 4 have the local dynamics and metapopulation for cycle 2. Fig. 5 and Fig. 6 have the local dynamics and metapopulation for cycle 3.

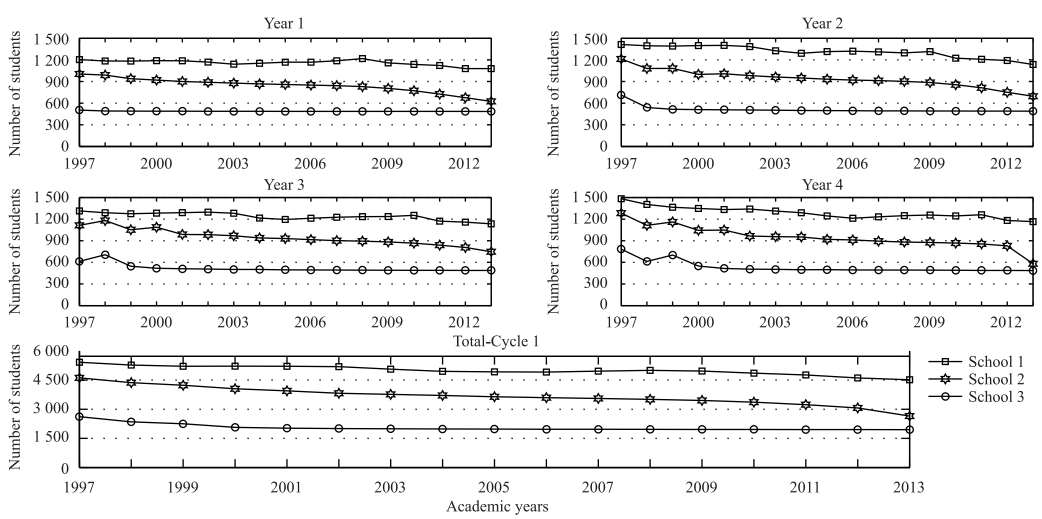

Fig. 1 Student population evolution from 1997 to 2013 for cycle 1 after local dynamics

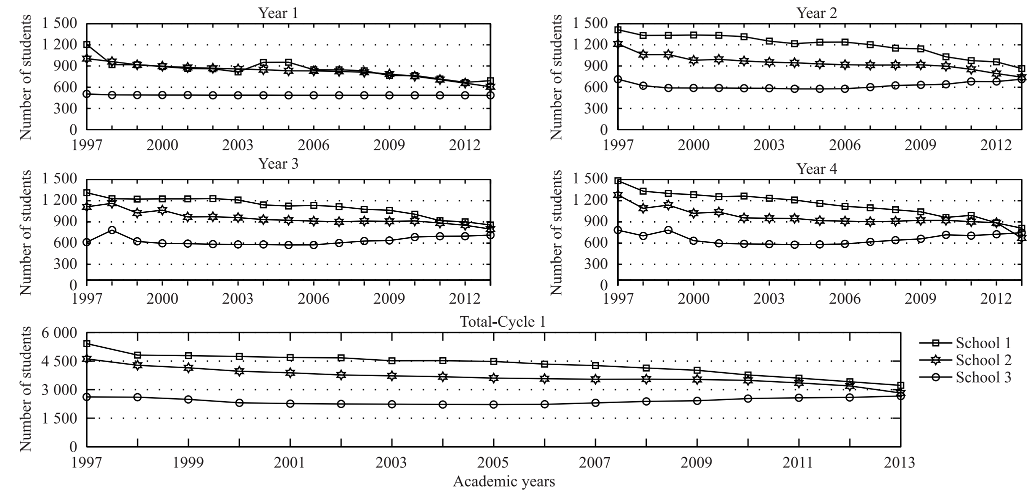

Fig. 2 Student population evolution from 1997 to 2013 for cycle 1 after metapopulation dynamics

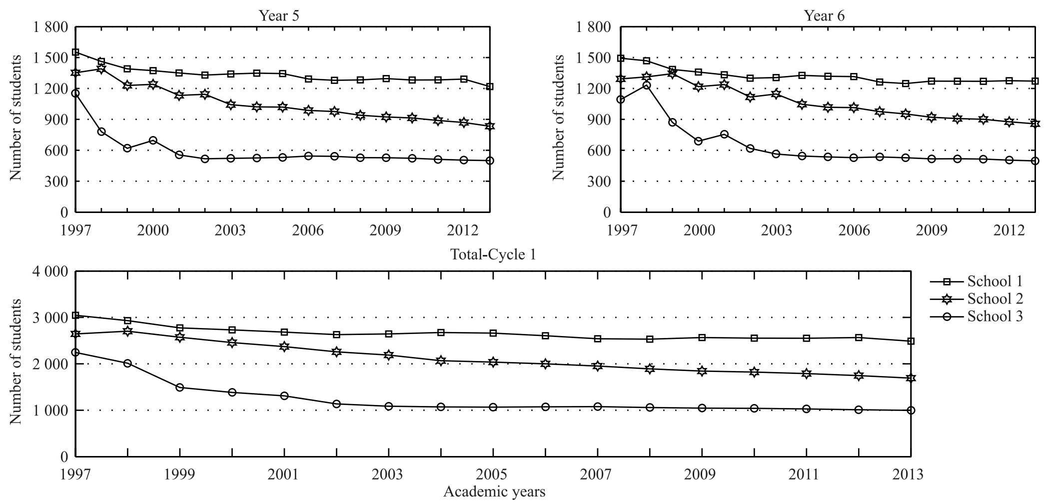

Fig. 3 Student population evolution from 1997 to 2013 for cycle 2 after local dynamics

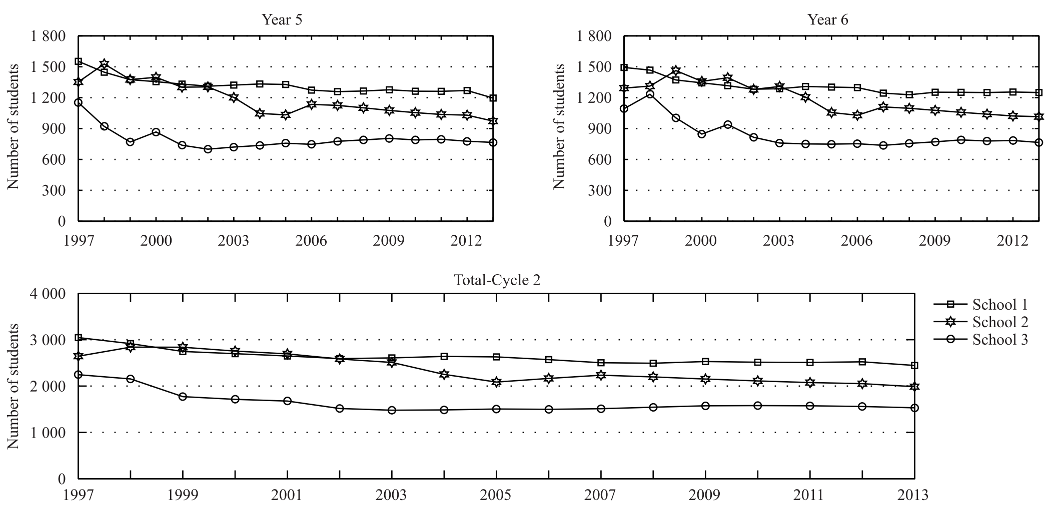

Fig. 4 Student population evolution from 1997 to 2013 for cycle 2 after metapopulation dynamics

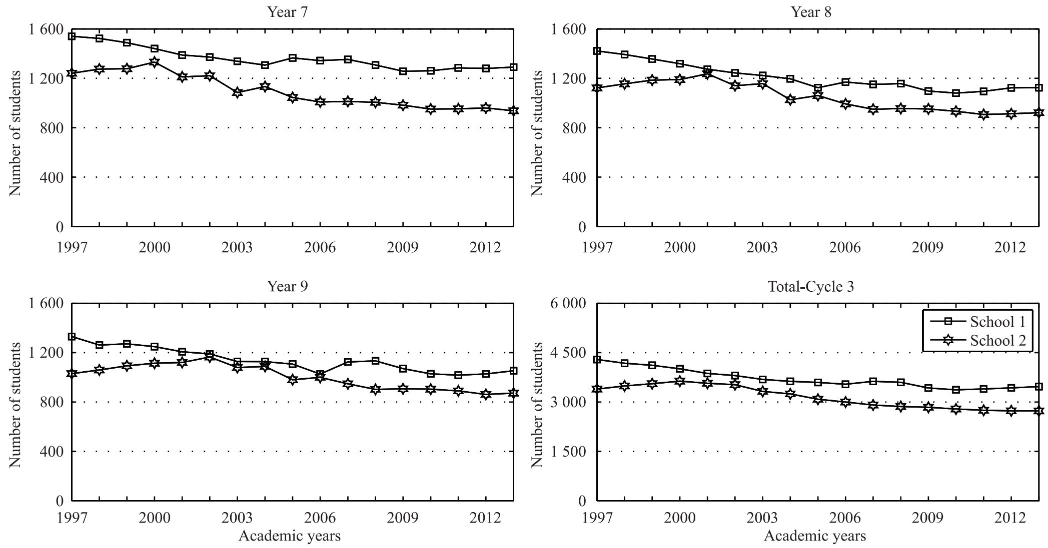

Fig. 5 Student population evolution from 1997 to 2013 for cycle 3 after local dynamics

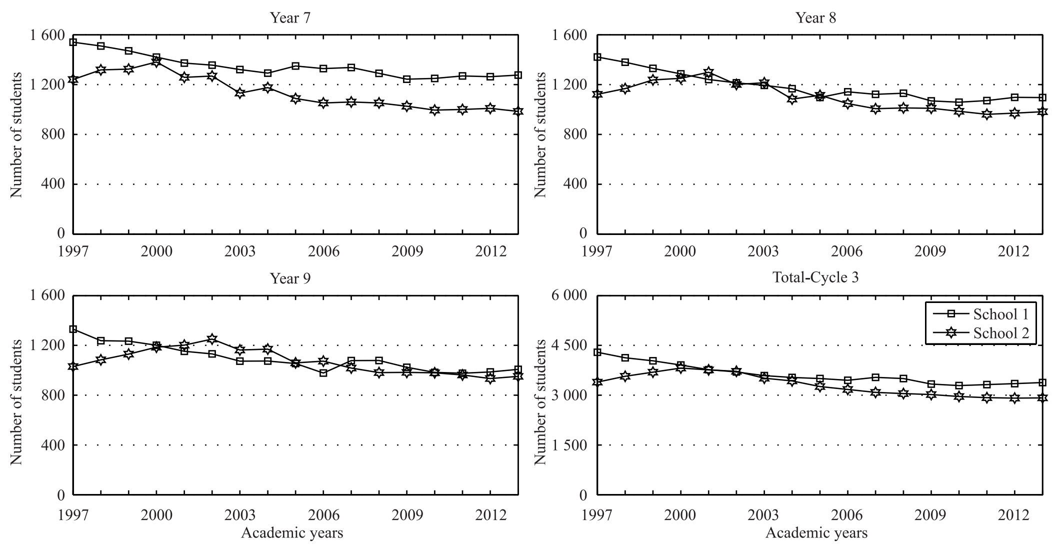

Fig. 6 Student population evolution from 1997 to 2013 for cycle 3 after metapopulation dynamics

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