Hindawi
Discrete Dynamics in Nature and Society
Volume 2018, Article ID 8152032, 18 pages
https://doi.org/10.1155/2018/8152032

Research Article

A Consistent Discrete Version of a Nonautonomous SIRVS Model

Joaquim Mateus,[1] César M. Silva ,[2] and S. Vaz[2]

[1] *Unidade de Investigação para o Desenvolvimento do Interior (UDI), Instituto Politécnico da Guarda, 6300-559 Guarda, Portugal*
[2] *Centro de Matemática e Aplicações e Departamento de Matemática, Universidade da Beira Interior, 6201-001 Covilhã, Portugal*

Correspondence should be addressed to César M. Silva; csilva@ubi.pt

Received 24 February 2018; Revised 27 May 2018; Accepted 29 May 2018; Published 24 June 2018

Academic Editor: Pilar R. Gordoa

A family of discrete nonautonomous SIRVS models with general incidence is obtained from a continuous family of models by applying Mickens nonstandard discretization method. Conditions for the permanence and extinction of the disease and the stability of disease-free solutions are determined. Concerning extinction and persistence, the consistency of those discrete models with the corresponding continuous model is discussed: if the time step is sufficiently small, when we have extinction (permanence) for the continuous model, we also have extinction (permanence) for the corresponding discrete model. Some numerical simulations are carried out to compare the different possible discretizations of our continuous model using real data.

1. Introduction

Most of the epidemiological models in the literature are continuous models. In spite of this, recently, there has been a growing interest in discrete-time models [1–7]. In this work, we will use Mickens nonstandard difference (NSFD) scheme to achieve a discretization of a family of continuous epidemiological models with vaccination and general incidence function considered in [8]. We have multiple objectives: firstly, we want to obtain conditions for extinction and permanence of the disease for the discrete family; next, having a continuous and a corresponding discrete family of models, we wish to discuss the problem of consistency of the discrete models with the corresponding continuous ones; finally, we intend to present some simulation results.

The dynamical consistency of a numerical scheme with the associated continuous system is not a precise definition. By the expression "dynamical consistency," it is meant that the numerical solutions replicate some of the properties of the continuous systems solutions.

We will consider the dynamical consistency regarding the permanence and extinction of the disease: whenever there is extinction (permanence) of the disease for the continuous-time model, the same holds for the discrete-time one. Several papers [9–15] discuss the dynamical consistency with respect to some particular properties of discrete epidemiological models obtained from continuous models by some NSFD scheme [16]. We note that while the papers cited above consider autonomous models, in the present work we discuss dynamical consistency for a nonautonomous model. To the best of our knowledge, this is the first work where the consistency of a discretized epidemiological model with the original continuous model is discussed in the nonautonomous context.

We remark that, even in the very particular case of autonomous models with mass action incidence and assuming that there is no disease related death, it follows from results in [17] that, when $\mathcal{R}_0 > 1$, the continuous autonomous model has one or two endemic equilibrium points coexisting with the disease-free equilibrium. The existence and stability of the equilibriums are governed by $\mathcal{R}_0$ and three additional thresholds: $\mathcal{R}_q$, $\mathcal{R}_v$, and $\mathcal{R}_{qv}$. These thresholds determine the qualitative behaviour of the system. This very particular situation shows that, even in the autonomous context, the qualitative behaviour can be difficult to determine.

One of the motivations for our work was that the difficulties increase considerably when we deal with a general nonautonomous situation. Thus, we decided to discuss an aspect of consistency that, nevertheless, is very important from the point of view of biomathematics: can we obtain consistency between continuous and discrete-time models from the point of view of persistence and extinction of the disease?

Regarding our simulation results, we considered two different types of computational experiments. Our first set of simulation results are designed to compare different possible discretizations of our continuous models. After this discussion, we apply our model to a real situation, considering data from the incidence of measles in France in the period 2012-2016. To the possible extent, this data is used to estimate our model parameters and the computational results obtained are compared with real data.

The law of mass action states that the rate of change in the disease incidence is directly proportional to the product of the number of susceptible and infective individuals and was the paradigm in the classic models in epidemiology. This is why classical models usually consider a bilinear incidence rate βSI, where S and I denote, respectively, the number of susceptible and infective individuals, to model the disease transmission. In spite of this, it is sometimes important to consider other forms of incidence functions. Another usual assumption is the time independence of the parameters model parameters: in fact, the majority of the epidemiological models in the literature are given by a system of autonomous differential or difference equations. Nevertheless, the assumption that the parameters are independent of time is not very realistic in many situations and it is useful to consider nonautonomous models that, for instance, allow the discussion of environmental and demographic effects that change with time [18, 19]. In this work the family of models considered is nonautonomous and the incidence rates are taken from a large class of functions.

Our model generalizes one obtained by Mickens nonstandard finite difference method from the continuous model [8] (see Section 2). In [20], a discrete nonautonomous epidemic model with vaccination and mass action incidence was obtained by Mickens method. We emphasize that, in the particular mass action case, our model is not exactly similar to the model in [20], although Mickens rules were considered in both. We briefly compare computationally these two slightly different models in Section 5.

The model we will consider is the following:

$$\begin{aligned} S_{n+1} - S_n &= \Lambda_n - \beta_n \varphi(S_{n+1}, I_n) - (\mu_n + p_n) S_{n+1} \\ &\quad + \eta_n V_{n+1} \\ I_{n+1} - I_n &= \beta_n \varphi(S_{n+1}, I_n) + \sigma_n \psi(V_{n+1}, I_n) \\ &\quad - (\mu_n + \alpha_n + \gamma_n) I_{n+1} \\ R_{n+1} - R_n &= \gamma_n I_{n+1} - \mu_n R_{n+1} \\ V_{n+1} - V_n &= p_n S_{n+1} - (\mu_n + \eta_n) V_{n+1} - \sigma_n \psi(V_{n+1}, I_n), \end{aligned} \quad (1)$$

$n \in \mathbb{N}$, where the classes S, I, R, and V correspond, respectively, to susceptible, infective, recovered, and vaccinated individuals and the parameter functions have the following meanings: Λ_n denotes the inflow of newborns in the susceptible class; the function $\beta_n\varphi$ is the incidence (into the infective class) from the susceptible individuals; the function $\sigma_n\psi$ is the incidence (into the infective class) from the vaccinated individuals; μ_n are the natural deaths; p_n represents the susceptibles vaccination; η_n represents the immunity loss and consequence influx in the susceptible class; α_n are the deaths occurring in the infective class; γ_n is the recovery. We will assume that (Λ_n), (μ_n), (p_n), (η_n), (α_n), (β_n), (σ_n), and (γ_n) are bounded and nonnegative sequences and that there are positive constants w_μ, w_Λ, w_p, k_φ, and k_ψ such that

(H1) the functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ are nonnegative and differentiable in $(\mathbb{R}_0^+)^2$ and the functions $\mathbb{R}_0^+ \ni x \to \partial_2\varphi(x,0)$ and $\mathbb{R}_0^+ \ni x \to \partial_2\psi(x,0)$ are nondecreasing and Lipschitz, with Lipschitz constants k_φ and k_ψ,

(H2) we have $\varphi(x,0) = \psi(x,0) = \varphi(0,y) = \psi(0,y) = 0$ for all $x, y \in \mathbb{R}_0^+$,

(H3) $\limsup_{n\to+\infty} \prod_{k=n}^{n+\omega_\mu} (1/(1+\mu_k)) < 1$,

(H4) $\liminf_{n\to+\infty} \sum_{k=n+1}^{n+\omega_\Lambda} \Lambda_k > 0$ and $\liminf_{n\to+\infty} \sum_{k=n+1}^{n+\omega_p} p_k > 0$,

(H5) functions $\mathbb{R}^+ \ni y \mapsto \phi(x,y)/y$ and $\mathbb{R}^+ \ni y \mapsto \psi(x,y)/y$ are nonincreasing.

In this work, we prove, when our conditions prescribe extinction (permanence) for the continuous model we also have extinction (permanence) for the corresponding discrete model as long as the time step is smaller than some constant (that depends on some model parameters and on the threshold condition). We also consider a family of examples of the periodic system of period 1 such that the continuous and the discrete-time system with time step $h = 1/L$ is not consistent, highlighting the importance of knowing that for time steps smaller than some explicit value we have consistency.

2. Discretization of the Continuous Model

We start with a nonautonomous SIRVS model that is slightly less general than the one considered in [8] and generalizes the one in [21]. Namely, we consider the model:

$$\begin{aligned} S' &= \Lambda(t) - \beta(t)\varphi(S) I - (\mu(t) + p(t)) S + \eta(t) V \\ I' &= [\beta(t)\varphi(S) + \sigma(t)\psi(V) - \mu(t) - \alpha(t) - \gamma(t)] I \\ R' &= \gamma(t) I - \mu(t) R \\ V' &= p(t) S - (\mu(t) + \eta(t)) V - \sigma(t)\psi(V) I. \end{aligned} \quad (2)$$

We assume that the functions $\Lambda, \mu, p, \eta, \alpha, \beta, \sigma$, and γ belong to the class $C^1(\mathbb{R}_0^+)$ and are nonnegative and bounded. We also require that

(C1) the functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are nonnegative, nondecreasing, differentiable, and Lipschitz with Lipschitz constants k_φ and k_ψ, respectively;

(C2) $\varphi(0) = \psi(0) = 0$;

(C3) there is $\omega > 0$ such that $\liminf_{t\to+\infty} \int_t^{t+\omega} \mu(s)ds > 0$.

In order to obtain threshold conditions for model (2), it was considered in [8] the following auxiliary system:

$$x' = \Lambda(t) - [\mu(t) + p(t)]x + \eta(t)y$$
$$y' = p(t)x - [\mu(t) + \eta(t)]y. \tag{3}$$

And for each solution $(x^*(t), y^*(t))$ of (3) with positive initial conditions, it was shown that the numbers

$$R_C^\ell(\lambda) = \liminf_{t\to\infty} \int_t^{t+\lambda} \beta(s)\varphi(x^*(s)) + \sigma(s)\psi(y^*(s)) - \mu(s) - \alpha(s) - \gamma(s)\,ds,$$
$$R_C^u(\lambda) = \limsup_{t\to\infty} \int_t^{t+\lambda} \beta(s)\varphi(x^*(s)) + \sigma(s)\psi(y^*(s)) - \mu(s) - \alpha(s) - \gamma(s)\,ds \tag{4}$$

are independent of the particular solution.

Using the above numbers, the following results are contained in results obtained in [8].

Theorem 1 (Theorem 1 of [8]). *Assume that conditions (C1), (C2), and (C3) hold. Then, if there is a constant $\lambda > 0$ such that $R_C^\ell(\lambda) > 0$, then the infectives I are permanent in system (2).*

Theorem 2 (Theorem 2 of [8]). *Assume that conditions (C1), (C2), and (C3) hold. Then if there is a constant $\lambda > 0$ such that $R_C^u(\lambda) < 0$, then the infectives I go to extinction in system (2).*

In the literature, several models were discretized using Mickens NSFD schemes [22–35]. Next, we will apply Micken's nonstandard method to obtain a discrete version of system (2).

Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}$ be a positive continuous function such that

$$\lim_{h\to 0} \phi(h) = 0. \tag{5}$$

Given $h \in \mathbb{R}^+$, we let $t = nh$, with $n \in \mathbb{N}$, and identify $S'(t)$ with

$$\frac{S(nh+h) - S(nh)}{\phi(h)}. \tag{6}$$

After deciding a nonlocal representation for the incidence function and that terms that do not correspond to an interaction will be considered in the $n+1$ time, the first equation in (2) becomes

$$\begin{aligned} S((n+1)h) - S(nh) = \phi(h)\big[\Lambda(nh) &- \beta(nh)\varphi(S((n+1)h))I(nh) \\ &- (\mu(nh) + p(nh))S((n+1)h) \\ &+ \eta(nh)V(nh+h)\big]. \end{aligned} \tag{7}$$

Writing $S_n = S(nh)$, $I_n = I(nh)$, $V_n = V(nh)$, $\Lambda_n = \phi(h)\Lambda(nh)$, $\beta_n = \phi(h)\beta(nh)$, $\mu_n = \phi(h)\mu(nh)$, $p_n = \phi(h)p(nh)$, and $\eta_n = \phi(h)\eta(nh)$, we have

$$S_{n+1} - S_n = \Lambda_n - \beta_n\varphi(S_{n+1})I_n - (\mu_n + p_n)S_{n+1} + \eta_n V_{n+1}. \tag{8}$$

Proceeding similarly for the other equations, we obtain the following discrete model:

$$\begin{aligned} S_{n+1} - S_n &= \Lambda_n - \beta_n\varphi(S_{n+1})I_n - (\mu_n + p_n)S_{n+1} + \eta_n V_{n+1} \\ I_{n+1} - I_n &= \beta_n\varphi(S_{n+1})I_n + \sigma_n\psi(V_{n+1})I_n - (\mu_n + \alpha_n + \gamma_n)I_{n+1} \\ R_{n+1} - R_n &= \gamma_n I_{n+1} - \mu_n R_{n+1} \\ V_{n+1} - V_n &= p_n S_{n+1} - (\mu_n + \eta_n)V_{n+1} - \sigma_n\psi(V_{n+1})I_n, \end{aligned} \tag{9}$$

$n \in \mathbb{N}_0$. We will consider a model that contains this one to obtain some of our results. Namely, based on model (9), in Sections 3 and 4 we will study model (1) that has a more general form for the incidence function.

Now, we need to make some definitions. We say that

(i) the infectives (I_n) are *permanent* if for any solution (S_n, I_n, R_n, V_n) of (1) with initial conditions $S_0, I_0, R_0, V_0 > 0$ there are constants $0 < m < M$ such that

$$m < \liminf_{n\to\infty} I_n \leqslant \limsup_{n\to\infty} I_n < M; \tag{10}$$

(ii) the infectives (I_n) go to *extinction* if for any solution (S_n, I_n, R_n, V_n) of (1) with initial conditions $S_0, I_0, R_0, V_0 \geqslant 0$ we have $\lim_{n\to\infty} I_n = 0$.

Similar definitions can be made for the other compartments. For instance, if there exist constants $0 < m < M$ such that for any solution (S_n, I_n, R_n, V_n) of (1) with initial conditions $S_0, I_0, R_0, V_0 > 0$ we have

$$m < \liminf_{n\to\infty} S_n \leqslant \limsup_{n\to\infty} S_n < M \tag{11}$$

we say that the susceptibles are permanent.

3. Permanence and Extinction in the Discrete Model

In this section, we will extend the results obtained for the model with the usual mass action incidence in [20] to our generalized family of models. Namely, suitable thresholds are defined and conditions for persistence and extinction of the disease are obtained. As a corollary of our results, we consider the periodic case where we have a unique number that establishes the boundary between the regions of permanence and extinction. Although the proofs of our results are inspired in [20], some difficulties must be dealt with. In particular, it was necessary to understand the right conditions to impose to the incidence functions in order to overcome the technical difficulties.

To lighten the reading, the proofs of our results are presented in the appendix.

3.1. Auxiliary Results. Consider the auxiliary system,

$$x_{n+1} = \frac{\Lambda_n + \eta_n y_{n+1} + x_n}{1 + \mu_n + p_n}$$
$$y_{n+1} = \frac{p_n x_{n+1} + y_n}{1 + \mu_n + \eta_n}. \quad (12)$$

Note that the auxiliary system describes the behaviour of the system in the absence of infection. If (Λ_n), (μ_n), (p_n), (η_n), (α_n), (μ_n), (σ_n), and (β_n) are constant sequences, then the linear system (12) becomes autonomous and corresponds to the linearization of the equations for (S_n) and (V_n) in the classical (autonomous) SIRVS model.

In order to proceed we need to recall some notions. A solution (u_n) of some system of difference equations $u_{n+1} = f_n(u_n)$ is said to be *attractive* if for all $n_0 \in \mathbb{N}$ and all $\varepsilon > 0$ there is $\sigma(n_0) > 0$ and $T(\varepsilon, n_0, u_0) \in \mathbb{N}$ such that if $(\overline{u}_n)$ is a solution with $\|u_0 - \overline{u}_0\| < \sigma(n_0)$ then $\|u_n - \overline{u}_n\| < \varepsilon$, for all $n \geqslant n_0 + T(\varepsilon, n_0, u_0)$. Additionally, if some solution is attractive and we can take T to be only dependent on ε, we say that it is *uniformly attractive*.

The following theorem furnishes some simple properties of system (12).

Lemma 3 (lemma 2.2 of [20]). *Assume that conditions (H3) and (H4) hold. Then*

(i) *all solutions (x_n, y_n) of system (12) with initial condition $x_0 \geqslant 0$ and $y_0 \geqslant 0$ are nonnegative for all $n \in \mathbb{N}_0$;*

(ii) *each fixed solution (x_n, y_n) of (12) is bounded and globally uniformly attractive for all $n \in \mathbb{N}_0$;*

(iii) *if (x_n, y_n) is a solution of (12) and $(\tilde{x}_n, \tilde{y}_n)$ is a solution of the system*

$$x_{n+1} = \frac{\Lambda_n + \eta_n y_{n+1} + x_n + f_n}{1 + \mu_n + p_n}$$
$$y_{n+1} = \frac{p_n x_{n+1} + y_n + g_n}{1 + \mu_n + \eta_n}, \quad (13)$$

with $(\tilde{x}_0, \tilde{y}_0) = (x_0, y_0)$ then there is a constant $L > 0$, only depending on μ_n, satisfying

$$\sup_{n\in\mathbb{N}_0} \{|\tilde{x}_n - x_n| + |\tilde{y}_n - y_n|\} \leqslant L \sup_{n\in\mathbb{N}_0} (|f_n| + |g_n|); \quad (14)$$

(iv) *there exist constants $m, M > 0$ such that, for each solution (x_n, y_n) of (12), we have*

$$m \leqslant \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n \leqslant M,$$
$$m \leqslant \liminf_{n\to\infty} y_n \leqslant \limsup_{n\to\infty} y_n \leqslant M; \quad (15)$$

(v) *when system (12) is ω–periodic, it has a unique positive ω–periodic solution which is globally uniformly attractive.*

We have the following lemma.

Lemma 4. *Assume that condition (H5) holds. Then we have the following:*

(i) *all solutions (S_n, I_n, R_n, V_n) of (1) with nonnegative initial conditions are nonnegative for all $n \in \mathbb{N}_0$;*

(ii) *all solutions (S_n, I_n, R_n, V_n) of (1) with positive initial conditions are positive for all $n \in \mathbb{N}_0$;*

(iii) *there is a constant $M > 0$ such that, if (S_n, I_n, R_n, V_n) is a solution of (1) with nonnegative initial conditions, then*

$$\limsup_{n\to+\infty} S_n + I_n + R_n + V_n < M. \quad (16)$$

Proof. See the appendix. □

For each λ and each particular solution $\xi_n^* = (x_n^*, y_n^*)$ of (12) with $x_0^* > 0$ and $y_0^* > 0$ we define the numbers

$$\mathscr{R}_D^\ell(\xi^*, \lambda) = \liminf_{n\to\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \partial_2 \varphi(x_{k+1}^*, 0) + \sigma_k \partial_2 \psi(y_{k+1}^*, 0)}{1 + \mu_k + \alpha_k + \gamma_k}, \quad (17)$$

$$\mathscr{R}_D^u(\xi^*, \lambda) = \limsup_{n\to\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \partial_2 \varphi(x_{k+1}^*, 0) + \sigma_k \partial_2 \psi(y_{k+1}^*, 0)}{1 + \mu_k + \alpha_k + \gamma_k}, \quad (18)$$

where $\partial_i f$ denotes the partial derivative of f with respect to the i-th variable. Contrarily to what one could expect, the next lemma shows that the numbers above do not depend on the particular solution $\xi_n = (x_n, y_n)$ of (12) with $x_n(0) > 0$ and $y_n(0) > 0$.

Lemma 5. *Assume that (H1), (H3), and (H4) hold. If $(\xi_1^*)_n = ((x_1)_n^*, (y_1)_n^*)$ and $(\xi_2^*)_n = ((x_2)_n^*, (y_2)_n^*)$ are two solutions of (12) with $x_i^*(0) > 0$ and $y_i^*(0) > 0$, $i = 1, 2$, then*

$$\mathscr{R}_D^\ell(\xi_1^*, \lambda) = \mathscr{R}_D^\ell(\xi_2^*, \lambda),$$
$$\mathscr{R}_D^u(\xi_1^*, \lambda) = \mathscr{R}_D^u(\xi_2^*, \lambda). \quad (19)$$

Proof. See the Appendix. □

By Lemma 5 we can drop the dependence of the particular solution and simply write $\mathscr{R}_D^\ell(\lambda)$ and $\mathscr{R}_D^u(\lambda)$ instead of $\mathscr{R}_D^\ell(\xi^*, \lambda)$ an $\mathscr{R}_D^u(\xi^*, \lambda)$, respectively.

3.2. Extinction and Permanence. We have the following result about the extinction of the disease.

Theorem 6 (extinction of the disease). *Assume that conditions (H1) to (H5) hold. Then*

(a) *if there is a constant* $\lambda > 0$ *such that* $\mathcal{R}_D^u(\lambda) < 1$, *then the infectives* (I_n) *go to extinction;*

(b) *any solution* $(x_n^*, 0, 0, y_n^*)$, *where* (x_n^*, y_n^*) *is a particular solution of system (12), is globally uniformly attractive.*

Proof. See the Appendix. □

We have the following result about the permanence of the disease.

Theorem 7 (permanence of the disease). *Assume that conditions (H1) to (H5) hold. If there is a constant* $\lambda > 0$ *such that* $\mathcal{R}_D^\ell(\lambda) > 1$ *then the infectives* (I_n) *are permanent in system (1).*

Proof. See the Appendix. □

We consider now the particular periodic case: assume that all parameters of system (1) are periodic with period $\omega \in \mathbb{N}$. By (v) in Lemma 3, there is an ω-periodic disease-free solution of (12), $\xi^* = (x_n^*, y_n^*)_{n\in\mathbb{N}}$. Thus, in the periodic setting, (17) and (18) become both equal to

$$\mathcal{R}_D^{per}(\xi^*) = \prod_{k=0}^{\omega-1} \frac{1 + \beta_k \partial_2 \varphi(x_{k+1}^*, 0) + \sigma_k \partial_2 \psi(y_{k+1}^*, 0)}{1 + \mu_k + \alpha_k + \gamma_k}. \quad (20)$$

Therefore, we obtain the corollary.

Corollary 8 (periodic case). *Assume that all coefficients are* ω*-periodic in (1) and that conditions (H1) to (H5) hold. Then*

(a) *if* $\mathcal{R}_D^{per}(\xi^*) < 1$ *then the infectives* (I_n) *go to extinction;*

(b) *the disease-free solution* $(x_n^*, 0, 0, y_n^*)$, *where* $(x_n^*, y_n^*)_{n\in\mathbb{N}}$ *is an disease-free* ω*-periodic solution of (12), is globally attractive;*

(c) *if* $\mathcal{R}_D^{per}(\xi^*) > 1$, *then the infectives* (I_n) *are permanent.*

Proof. See the Appendix. □

4. Consistency

In this section, under the additional assumption that the parameter functions Λ, μ, η, and p are constant, we will get a result stating that when our integral conditions prescribe extinction (persistence) for the continuous-time model, then the discrete-time conditions prescribe extinction (persistence) for the corresponding discrete-time models, as long as the time step is less than some constant. Throughout this section, we assume that the parameter functions Λ, μ, η, and p are constant functions and $\phi(h)$ will be the function used in the discretization of the derivative.

We consider the continuous-time model (2) and, for a given time step h, the corresponding discrete-time model, that is, the discrete-time model with parameters $\beta_k^h = \phi(h)\beta(kh)$, $\sigma_k^h = \phi(h)\sigma(kh)$, $\Lambda_k^h = \phi(h)\Lambda$, $\mu_k^h = \phi(h)\mu$, $p_k^h = \phi(h)p$, $\eta_k^h = \phi(h)\eta$, $\alpha_k^h = \phi(h)\alpha(kh)$, and $\gamma_k^h = \phi(h)\gamma(kh)$.

For a given time step $h > 0$, the expressions $\mathcal{R}_D^\ell(\lambda)$ and $\mathcal{R}_D^u(\lambda)$ in (17) and (18) become, in our context,

$$\mathcal{R}_D^\ell(\lambda, h) = \liminf_{n\to\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k^h \varphi(x_{k+1}^*) + \sigma_k^h \psi(y_{k+1}^*)}{1 + \mu_k^h + \alpha_k^h + \gamma_k^h},$$
$$\mathcal{R}_D^u(\lambda, h) = \limsup_{n\to\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k^h \varphi(x_{k+1}^*) + \sigma_k^h \psi(y_{k+1}^*)}{1 + \mu_k^h + \alpha_k^h + \gamma_k^h}, \quad (21)$$

where (x_k^*, y_k^*) is the solution of the (in our context autonomous) system (12).

We have the following result.

Theorem 9. *For system (2), assume that* $\Lambda(t) = \Lambda$, $\mu(t) = \mu$, $\eta(t) = \eta$, *and* $p(t) = p$ *for all* $t \geqslant 0$ *and that the functions* $\alpha(t)$, $\gamma(t)$, $\beta(t)$, *and* $\sigma(t)$ *are differentiable, nonnegative, and bounded and have bounded derivative. Assume also that conditions (C1) to (C3) hold and let*

$$h_{max}^u = -\frac{R_C^u(\lambda)}{\sup_{t\geqslant 0} |f'(t)| (\lambda + 1)},$$
$$h_{max}^\ell = \frac{R_C^\ell(\lambda)}{\sup_{t\geqslant 0} |f'(t)| (\lambda + 1)}, \quad (22)$$

where

$$f(t) = \beta(t)\varphi\left(\frac{\Lambda(\mu+\eta)}{\mu(\mu+\eta+p)}\right) + \sigma(t)\psi\left(\frac{p\Lambda}{\mu(\mu+\eta+p)}\right) - \mu - \alpha(t) - \gamma(t). \quad (23)$$

Then,

(a) *if* $\mathcal{R}_C^u(\lambda) < 0$, *then* $\mathcal{R}_D^u(\lfloor\lambda/h\rfloor, h) < 1$ *for all* $h \in]0, h_{max}^u[$;

(b) *if* $\mathcal{R}_C^\ell(\lambda) > 0$, *then* $\mathcal{R}_D^\ell(\lfloor\lambda/h\rfloor, h) > 1$ *for all for all* $h \in]0, h_{max}^\ell[$.

Proof. Observe that $(x_n, y_n) = (a, b)$, $n \in \mathbb{N}$ and $(x(t), y(t)) = (a, b)$, $t \in \mathbb{R}$, where

$$(a, b) = \left(\frac{\Lambda(\mu+\eta)}{[\mu(\mu+\eta+p)]}, \frac{p\Lambda}{[\mu(\mu+\eta+p)]}\right), \quad (24)$$

are, respectively, solutions of system (12) and system (3). Thus,

$$\mathcal{R}_C^u(\lambda) = \limsup_{t\to\infty} \int_t^{t+\lambda} \beta(s)\varphi(a) + \sigma(s)\psi(b) - [\mu(s) + \alpha(s) + \gamma(s)]\, ds,$$
$$\mathcal{R}_D^u\left(\left\lfloor\frac{\lambda}{h}\right\rfloor, h\right) = \limsup_{n\to\infty} \prod_{k=n}^{n+\lfloor\lambda/h\rfloor} \frac{1 + \beta_k^h \varphi(a) + \sigma_k^h \psi(b)}{1 + \mu_k^h + \alpha_k^h + \gamma_k^h}. \quad (25)$$

By contradiction, assume that

$$\mathscr{R}_C^u(\lambda) < 0, \tag{26}$$

and that there is a sequence $(h_m)_{m\in\mathbb{N}}$ such that $h_m \to 0$ as $m \to +\infty$ and

$$\mathscr{R}_D^u\left(\left\lfloor \frac{\lambda}{h_m}\right\rfloor, h_m\right) = \limsup_{n\to\infty} \prod_{k=n}^{n+\lfloor \lambda/h_m\rfloor} \frac{1+\beta_k^{h_m}\varphi(a)+\sigma_k^{h_m}\psi(b)}{1+\mu_k^{h_m}+\alpha_k^{h_m}+\gamma_k^{h_m}} \geqslant 1, \tag{27}$$

for all $m \in \mathbb{N}$. By (27), we conclude that, for each $m \in \mathbb{N}$, there are sequences $(h_m)_{m\in\mathbb{N}}$ and $(n_{m,r})_{r\in\mathbb{N}}$ such that $h_m \to 0$ as $m \to +\infty$, $n_{m,r} \to +\infty$ as $r \to +\infty$ and

$$\prod_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m\rfloor} \left(1+\beta_k^{h_m}\varphi(a)+\sigma_k^{h_m}\psi(b)\right) > (1-h_m) \prod_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m\rfloor} \left(1+\mu_k^{h_m}+\alpha_k^{h_m}+\gamma_k^{h_m}\right). \tag{28}$$

By (28), we have

$$\sum_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m\rfloor} \left(\beta_k^{h_m}\varphi(a)+\sigma_k^{h_m}\psi(b)-\mu_k^{h_m}-\alpha_k^{h_m}-\gamma_k^{h_m}\right) > \frac{\left(B_{n_{m,r},\lambda,h_m}-A_{n_{m,r},\lambda,h_m}-C_{n_{m,r},\lambda,h_m}\right)}{h_m}, \tag{29}$$

where

$$A_{n,L,h} \coloneqq -h + h\prod_{k=n}^{n+\lfloor L/h\rfloor}\left(1+\beta_k^h\varphi(a)+\sigma_k^h\psi(b)\right) - h\sum_{k=n}^{n+\lfloor L/h\rfloor}\left(\beta_k^h\varphi(a)+\sigma_k^h\psi(b)\right), \tag{30}$$

$$B_{n,L,h} \coloneqq -h + h\prod_{k=n}^{n+\lfloor L/h\rfloor}\left(1+\mu_k^h+\alpha_k^h+\gamma_k^h\right) - h\sum_{k=n}^{n+\lfloor L/h\rfloor}\left(\mu_k^h+\alpha_k^h+\gamma_k^h\right), \tag{31}$$

$$C_{n,L,h} = h\prod_{k=n}^{n+\lfloor L/h\rfloor}\left(1+\mu_k^h+\alpha_k^h+\gamma_k^h\right). \tag{32}$$

and, multiplying both sides by h_m, we get

$$\phi(h_m)\sum_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m\rfloor} h_m\left[\beta(kh_m)\varphi(a)+\sigma(kh_m)\psi(b) - \mu(kh_m)-\alpha(kh_m)-\gamma(kh_m)\right] > B_{n_{m,r},\lambda,h_m} - A_{n_{m,r},\lambda,h_m} - C_{n_{m,r},\lambda,h_m}. \tag{33}$$

We also have

$$\begin{aligned}\left|A_{n_{m,r},\lambda,h_m}\right| &\leqslant h_m\sum_{k=2}^{\lfloor \lambda/h_m\rfloor}\binom{\lfloor \lambda/h_m\rfloor}{k}\left[(\beta^u\varphi(a)+\sigma^u\psi(b))\right]^k\left[\phi(h_m)\right]^k \\ &\leqslant h_m\sum_{k=0}^{\lfloor \lambda/h_m\rfloor}\binom{\lfloor \lambda/h_m\rfloor}{k}\left[(\beta^u\varphi(a)+\sigma^u\psi(b))\right]^k\left[\phi(h_m)\right]^k \\ &= h_m\left[1+(\beta^u\varphi(a)+\sigma^u\psi(b))\phi(h_m)\right]^{\lfloor \lambda/h_m\rfloor}.\end{aligned} \tag{34}$$

Noting that, by (5), we have

$$\begin{aligned}\lim_{m\to+\infty}\left[1+(\beta^u\varphi(a)+\sigma^u\psi(b))\phi(h_m)\right]^{\lfloor \lambda/h_m\rfloor} &= \lim_{m\to+\infty}\left[\left(1+\frac{\beta^u\varphi(a)+\sigma^u\psi(b)}{1/\phi(h_m)}\right)^{1/\phi(h_m)}\right]^{\phi(h_m)\lfloor \lambda/h_m\rfloor} \\ &= e^{(\beta^u\varphi(a)+\sigma^u\psi(b))\lambda}\end{aligned} \tag{35}$$

and that a convergent sequence is bounded; by (34) there is $C_1 > 0$ such that

$$\left|A_{n_{m,r},\lambda,h_m}\right| \leqslant C_1 h_m. \tag{36}$$

Similarly, we have

$$\begin{aligned}\left|B_{n_{m,r},\lambda,h_m}\right| &\leqslant \sum_{k=2}^{\lfloor \lambda/h_m\rfloor}\binom{\lfloor \lambda/h_m\rfloor}{k}\left[(\mu^u+\alpha^u+\gamma^u)\right]^k\left[\phi(h_m)\right]^k \\ &\leqslant \sum_{k=0}^{\lfloor \lambda/h_m\rfloor}\binom{\lfloor \lambda/h_m\rfloor}{k}\left[(\mu^u+\alpha^u+\gamma^u)\right]^k\left[\phi(h_m)\right]^k \\ &= h_m\left[1+(\mu^u+\alpha^u+\gamma^u)\phi(h_m)\right]^{\lfloor \lambda/h_m\rfloor}.\end{aligned} \tag{37}$$

Using (5) again, we get

$$\lim_{m\to+\infty}\left[1+(\mu^u+\alpha^u+\gamma^u)\phi(h_m)\right]^{\lfloor \lambda/h_m\rfloor} \leqslant e^{(\mu^u+\alpha^u+\gamma^u)\lambda}. \tag{38}$$

There is $C_2 > 0$ such that

$$\left|B_{n_{m,r},\lambda,h_m}\right| \leqslant C_2 h_m. \tag{39}$$

Finally, we have

$$\begin{aligned}\left|C_{n_{m,r},\lambda,h_m}\right| &= h_m\prod_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m\rfloor}\left(1+\mu_k^{h_m}+\alpha_k^{h_m}+\gamma_k^{h_m}\right) \\ &= h_m\left(1+3\phi(h_m)\max\{\mu^u,\alpha^u,\gamma^u\}\right)^{\lfloor \lambda/h_m\rfloor+1}.\end{aligned} \tag{40}$$

According to (5), we obtain

$$\begin{aligned}\lim_{m\to+\infty}&\left(1+3\phi(h_m)\max\{\mu^u,\alpha^u,\gamma^u\}\right)^{\lfloor\lambda/h_m\rfloor+1}\\&=\lim_{m\to+\infty}\left[\left(1+\frac{3\max\{\mu^u,\alpha^u,\gamma^u\}}{1/\phi(h_m)}\right)^{1/\phi(h_m)}\right]^{\phi(h_m)(\lfloor\lambda/h_m\rfloor+1)}\\&=e^{3\max\{\mu^u,\alpha^u,\gamma^u\}\lambda}.\end{aligned}\tag{41}$$

There is $C_3>0$ such that

$$\left|C_{n_{m,r},\lambda,h_m}\right|\leqslant C_3h_m.\tag{42}$$

Thus,

$$\begin{aligned}B_{n_{m,r},\lambda,h_m}&-A_{n_{m,r},\lambda,h_m}-C_{n_{m,r},\lambda,h_m}\\&\leqslant\left|A_{n_{m,r},\lambda,h_m}\right|+\left|B_{n_{m,r},\lambda,h_m}\right|+\left|C_{n_{m,r},\lambda,h_m}\right|\\&\leqslant(C_1+C_2+C_3)h_m,\end{aligned}\tag{43}$$

for all $m\geqslant M$. Since the right hand side of (43) is independent of $n_{m,r}$, we conclude that

$$B_{n_{m,r},\lambda,h_m}-A_{n_{m,r},\lambda,h_m}-C_{n_{m,r},\lambda,h_m}\longrightarrow 0,\tag{44}$$

as $m\to+\infty$, uniformly in r.

On the other hand, we note that the C^1 function $f:\mathbb{R}_0^+\to\mathbb{R}$ given by

$$f(t)=\beta(t)\varphi(a)+\sigma(t)\psi(b)-\mu(t)-\alpha(t)-\gamma(t)\tag{45}$$

is Riemann-integrable on any bounded interval $I\subset\mathbb{R}_0^+$.

We have that

$$\sum_{k=n_{m,r}}^{n_{m,r}+\lfloor\lambda/h_m\rfloor}h_mf(kh_m)+\left(\lambda-\left\lfloor\frac{\lambda}{h_m}\right\rfloor h_m\right)f\left(n_{m,r}h_m+\left\lfloor\frac{\lambda}{h_m}\right\rfloor h_m\right)\tag{46}$$

is a Riemann sum of

$$\int_{n_{m,r}h_m}^{n_{m,r}h_m+\lambda}\beta(s)\varphi(a)+\sigma(s)\psi(b)-[\mu(s)+\alpha(s)+\gamma(s)]\,ds\tag{47}$$

with respect to the partition

$$\left\{n_{m,r}h_m,n_{m,r}h_m+h_m,\ldots,n_{m,r}h_m+\left\lfloor\frac{\lambda}{h_m}\right\rfloor h_m,n_{m,r}h_m+\lambda\right\}\tag{48}$$

of size h_m of the interval $[n_{m,r},n_{m,r}+\lambda]$. Note that

$$\begin{aligned}s_{m,r}&:=\left(\lambda-\left\lfloor\frac{\lambda}{h_m}\right\rfloor h_m\right)f\left(n_{m,r}h_m+\left\lfloor\frac{\lambda}{h_m}\right\rfloor h_m\right)\\&\leqslant h_mf^u:=s_m\end{aligned}\tag{49}$$

and $s_m\to 0$ as $m\to+\infty$, uniformly in r.

Since f is C^1 with bounded derivative, for any $h>0$ we have

$$\left|f(x)-f(x+h)\right|\leqslant Ch,\tag{50}$$

where $C=\sup_{t\geqslant0}|f'(t)|$. We conclude that

$$\begin{aligned}&\left|\sum_{k=n_{m,r}}^{n_{m,r}+\lfloor\lambda/h_m\rfloor}h_mf(kh_m)+s_{m,r}-\int_{n_{m,r}h_m}^{n_{m,r}h_m+\lambda}f(s)\,ds\right|\\&\quad<Ch_m^2\left\lfloor\frac{\lambda}{h_m}\right\rfloor+Ch_m^2<C(\lambda+1)h_m,\end{aligned}\tag{51}$$

thus

$$\begin{aligned}\sum_{k=n_{m,r}}^{n_{m,r}+\lfloor\lambda/h_m\rfloor}h_mf(kh_m)&<\int_{n_{m,r}h_m}^{n_{m,r}h_m+\lambda}f(s)\,ds-s_{m,r}\\&\quad+C(\lambda+1)h_m,\end{aligned}\tag{52}$$

and therefore

$$\begin{aligned}&\sum_{k=n_{m,r}}^{n_{m,r}+\lfloor\lambda/h_m\rfloor}\phi(h_m)h_mf(kh_m)\\&\quad<\phi(h_m)\left[\int_{n_{m,r}h_m}^{n_{m,r}h_m+\lambda}f(s)\,ds+C(\lambda+1)h_m\right].\end{aligned}\tag{53}$$

By (53) we conclude that, given $\delta>0$, there is $r_m\in\mathbb{N}$ such that, for all $r\geqslant r_m$,

$$\begin{aligned}&\phi(h_m)\sum_{k=n_{m,r_m}}^{n_{m,r_m}+\lfloor\lambda/h_m\rfloor}h_mf(kh_m)\\&\quad<\phi(h_m)\left[\mathscr{R}_C^u(\lambda)+\delta+C(\lambda+1)h_m\right].\end{aligned}\tag{54}$$

Finally, recalling that $\mathscr{R}_C^u(\lambda)<0$, by assumption, by the arbitrariness of $\delta>0$ and the fact that $h_m\to0$ as $m\to+\infty$, we obtain for sufficiently large $m\in\mathbb{N}$,

$$0\leqslant\phi(h_m)\sum_{k=n_{m,r_m}}^{n_{m,r_m}+\lfloor\lambda/h_m\rfloor}h_mf(kh_m)<0,\tag{55}$$

which is a contradiction. We obtain (a).

A similar argument allows us to prove (b). In fact, assuming by contradiction that

$$\mathscr{R}_C^\ell(\lambda)>0,\tag{56}$$

and that there is a sequence $(h_m)_{m\in\mathbb{N}}$ such that $h_m \to 0$ as $m \to +\infty$ and

$$\begin{aligned}&\mathscr{R}_D^{\ell}\left(\left\lfloor \frac{\lambda}{h_m}\right\rfloor, h_m\right)\\ &\quad = \liminf_{n\to\infty} \prod_{k=n}^{n+\lfloor \lambda/h_m\rfloor} \frac{1+\beta_k^{h_m}\varphi(a)+\sigma_k^{h_m}\psi(b)}{1+\mu_k^{h_m}+\alpha_k^{h_m}+\gamma_k^{h_m}} \leqslant 1,\end{aligned} \tag{57}$$

it is possible to conclude that

$$\begin{aligned}&\phi(h_m)\sum_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m\rfloor} h_m\left(\beta(kh_m)\varphi(a)+\sigma(kh_m)\psi(b)\right.\\ &\quad \left.-\mu(kh_m)-\alpha(kh_m)-\gamma(kh_m)\right) < B_{n_{m,r},\lambda,h_m}\\ &\quad - A_{n_{m,r},\lambda,h_m} + C_{n_{m,r},\lambda,h_m},\end{aligned} \tag{58}$$

where $A_{n_{m,r},\lambda,h_m}$, $B_{n_{m,r},\lambda,h_m}$, and $C_{n_{m,r},\lambda,h_m}$ are given, respectively, by (30), (31), and (42) and still satisfy (36), (39), and (42). Consequently, given $\delta > 0$, there is $r_m \in \mathbb{N}$ such that, for all $r \geqslant r_m$,

$$\begin{aligned}&\phi(h_m)\sum_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m\rfloor} h_m f(kh_m)\\ &\quad > \phi(h_m)\left[\int_{n_{m,r}h_m}^{n_{m,r}h_m+\lambda} f(s)\,ds - \delta + C(\lambda+1)h_m\right].\end{aligned} \tag{59}$$

Recalling that $\mathscr{R}_C^{\ell}(\lambda) > 0$, by assumption and since $\delta > 0$ is arbitrary, we obtain for sufficiently large $m \in \mathbb{N}$,

$$0 \geqslant \phi(h_m)\sum_{k=n_{m,r_m}}^{n_{m,r_m}+\lfloor \lambda/h_m\rfloor} f(kh_m) > 0, \tag{60}$$

which is a contradiction. We obtain (b) and the theorem follows. □

Next, for each $L \in \mathbb{N}$, we give an example of a periodic system of period 1 such that the continuous and the discrete-time system with time step $h = 1/L$ are not consistent; namely, we will have persistence for the continuous-time model and extinction for the discrete-time model with time step $h = 1/L$.

Example 10. Let $L \in \mathbb{N}$. Consider in system (2) that $\phi(x) = \psi(x) = x$, that, with the exception of σ and β, all parameters are constant, that $\Lambda = \mu$, and that

$$\sigma(t) = \beta(t) = d\left[1 + c\sin^2(2\pi L t)(1+\cos(2\pi t))\right]. \tag{61}$$

We obtain a periodic system of period 1.

In this context, $(x_n, y_n) = (a, b)$, $n \in \mathbb{N}$, and $(x(t), y(t)) = (a, b)$, $t \in \mathbb{R}$, where

$$(a, b) = \left(\frac{\mu+\eta}{\mu+\eta+p}, \frac{p}{\mu+\eta+p}\right), \tag{62}$$

are, respectively, solutions of system (12) and system (3). It is now possible to compute the number $\mathscr{R}_C^{\ell}(1)$. In fact, noting that $x^*(t) + y^*(t) = 1$, we get

$$\begin{aligned}\mathscr{R}_C^{\ell}(1) &= \int_0^1 \beta(s)x^*(s) + \sigma(s)y^*(s) - \mu - \alpha - \gamma\,ds\\ &= \int_0^1 d\left[1 + c\sin^2(2\pi L t)(1+\cos(2\pi t))\right]ds\\ &\quad - \mu - \alpha - \gamma = d\left(1 + \frac{c}{2}\right) - \mu - \alpha - \gamma.\end{aligned} \tag{63}$$

We can also compute $\mathscr{R}_D^{\ell}(1, 1/L)$. Namely, we have

$$\mathscr{R}_D^{\ell}\left(1, \frac{1}{L}\right) = \frac{1 + d/L}{1+\mu+\alpha+\gamma}. \tag{64}$$

If we let d be sufficiently small so that $d < (\mu+\alpha+\gamma)L$, or in other words, $d < (\mu+\alpha+\gamma)$ and c be sufficiently large so that $c > (2/d)(\mu+\gamma+\alpha-d)$, we obtain

$$\begin{aligned}&\mathscr{R}_C^{\ell}(1) > 1 \Longleftrightarrow\\ &\frac{1 + d(1+c/2)}{1+\mu+\alpha+\gamma} > 1,\\ &\mathscr{R}_D^{\ell}\left(1, \frac{1}{L}\right) < 1 \Longleftrightarrow\\ &\frac{1+d/L}{1+\mu+\alpha+\gamma} < 1.\end{aligned} \tag{65}$$

So we conclude that we do not have consistency for time step $1/L$.

Let $L = 6$ and consider the continuous model with the following parameters $\mu = \Lambda = 0.25$, $\gamma = 0.3$, $\alpha = 0.05$, $\eta = 0.05$, $p = 2/3$, $d = 0.6$, and $c = 1.5$. In Figure 1, we plot function β (or similarly σ) and the component $I(t)$ of the solution of system (2) given by the solver of Mathematica® (that we take to represent the solution of the continuous-time model) and the solution of the discrete-time model (9) with time step 1/6. As can be seen, the infectives are persistent in the continuous-time model but go to extinction in the discrete-time model. We have inconsistency in this case.

Note that, changing $\beta(t)$ and $\sigma(t)$ slightly, we can construct an example of a periodic system with period 1 where the infectives in the continuous-time model go to extinction but, in the discrete-time model with time step $h = 1/L$, the infectives are persistent.

Furthermore, we emphasize that this lack of consistency is not a result of the discretization method used but simply a result of the fact that the time steps lead to a situation where the points n/L where the functions β and σ are evaluated (in order to obtain the discrete-time parameters) correspond to minimums of β and σ.

5. Simulation

Our objective in this section is twofold. On the one hand, we want to consider different incidence functions φ, corresponding to different discretizations of our continuous model, and

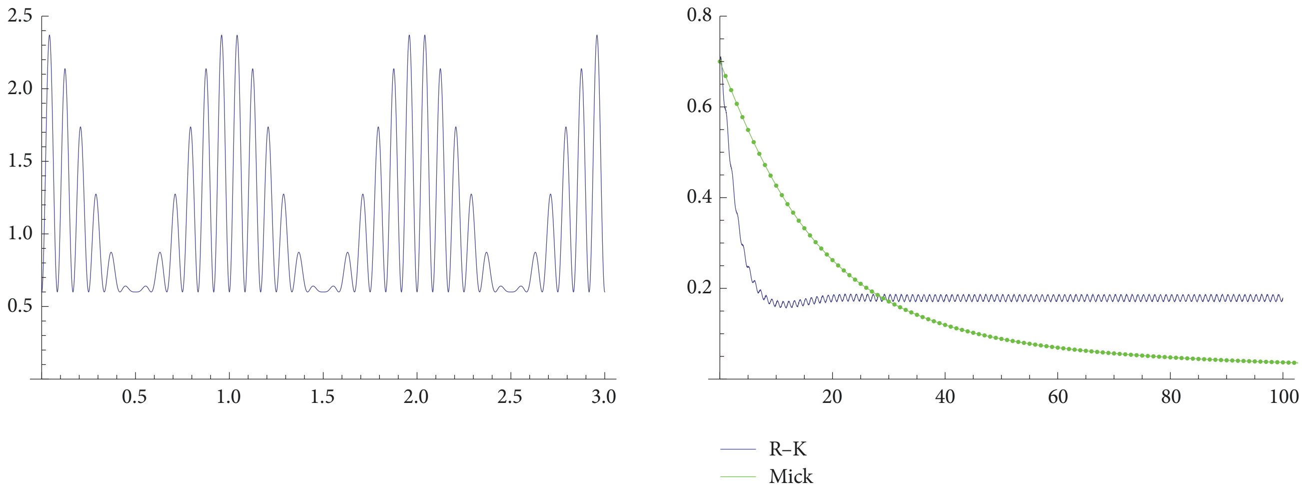

Figure 1: Left: function β; right: inconsistency (time step=1/6).

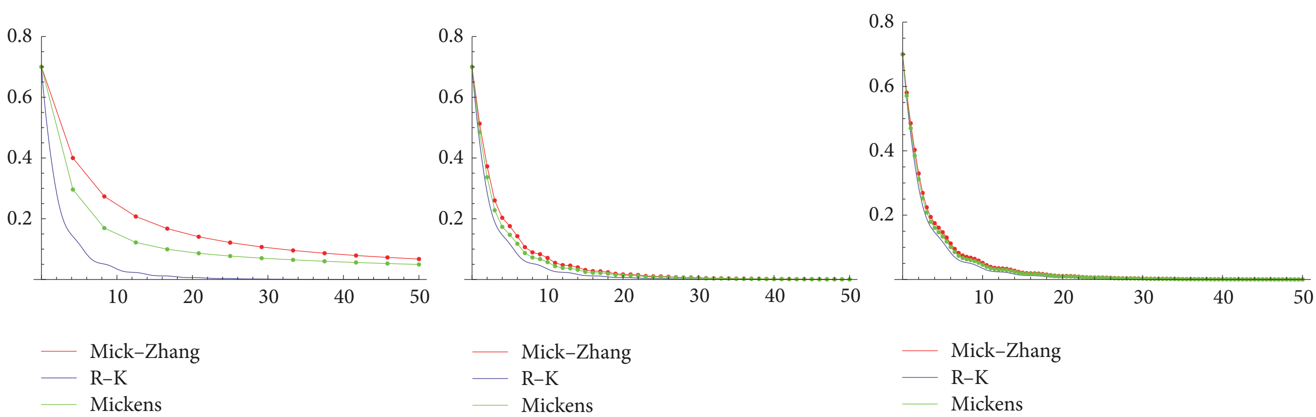

Figure 2: SI; $\phi(h) = h + 0.2h^2$; step-size: 4, 1, 0.5.

compare the several discrete models obtained. We do this in the first subsection. On the other hand, we want to use our model to describe a real situation. We do this in the second subsection where we consider data from the incidence of measles in France in the period 2012-2016.

5.1. Simulation with Several NSFD Schemes. In this subsection we do some simulation to illustrate our results. To begin, we compare our model (1) with mass action incidence ($\varphi(S, I) = SI$ and $\psi(V, I) = VI$) with Zhang's model [20]. We use the following set of parameters: $\phi(h) = h + 0.2h^2$, $\Lambda = 0.5$, $\mu(t) = \gamma(t) = \delta(t) = 0.3$, $\alpha(t) = 0.05$, $\eta = 0.05$, $p = 2/3$ and

$$\beta(t) = \sigma(t) = b\left(1 + 0.3\cos\left(\frac{t\pi}{2}\right)\right). \quad (66)$$

Setting $b = 0.3$ we obtain $\mathscr{R}_C^u(4) = -0.6 < 0$ and thus we conclude that we have extinction for the continuous model. Taking time steps equal to 4, 1, and 0.5, we get $\mathscr{R}_D^u(0, 4) = \mathscr{R}_D^\ell(0, 4) = 1$, $\mathscr{R}_D^u(3, 1) = 0.644 < 1$, and $\mathscr{R}_D(7, 0.5) = 0.601 < 1$ and we conclude that we have extinction for time steps 1 and 0.5. For these parameters, we have consistency in the sense of Theorem 9 as long as the time step is less than 0.05. Clearly, there is numerical evidence that there is consistency even for higher time steps. Figure 2 illustrates this situation.

Changing b to 0.9 we obtain $\mathscr{R}_C^\ell(4) = 3.4 > 0$ and thus we conclude that we have persistence. Taking time steps equal to 2, 1, and 0.5, we get $\mathscr{R}_D^\ell(1, 2) = 3.201 > 1$, $\mathscr{R}_D^\ell(3, 1) = 5.9 > 1$, and $\mathscr{R}_D^\ell(7, 0.5) = 10.2 > 1$ and we conclude that we have persistence for all these time steps. Figure 3 illustrates this situation. Figures 2 and 3 suggest that numerically our model is slightly better than Zhang's model, at least for large time steps.

Next, we compare our model with the discretized model obtained by Euler method and the output of the Mathematica solver ODE (that uses a Runge-Kutta method). Considering $b = 0.3$, we get extinction for the continuous-time model, as we already saw. Taking time steps equal to 2, 1, and 0.5, we can see in Figure 4 that for all methods considered and all time steps we have extinction, although the behaviour of our model shadows better the behaviour given by Mathematica's solver, at least for these time steps.

Changing b to 0.9 we already saw that we get persistence for the continuous model. Figure 5 illustrates this situation.

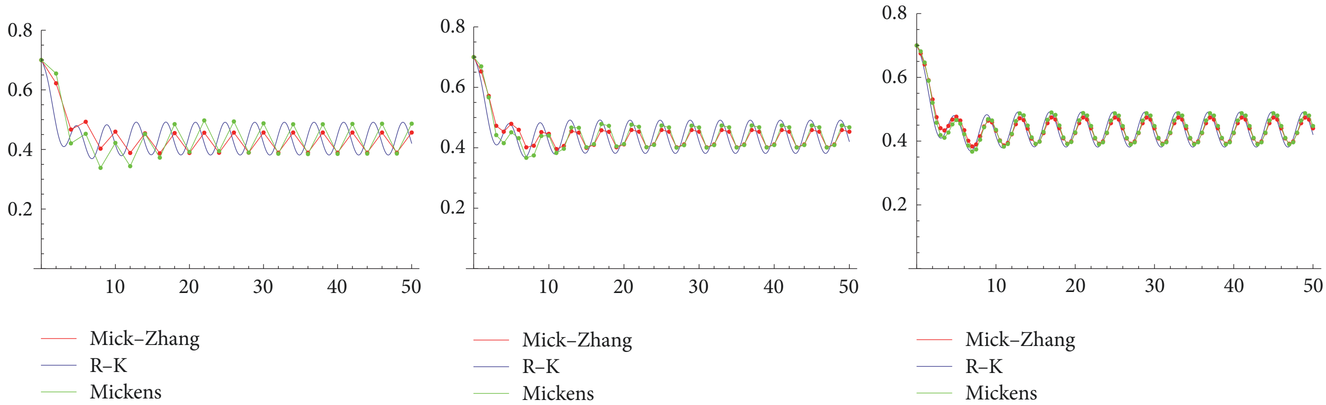

FIGURE 3: SI; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

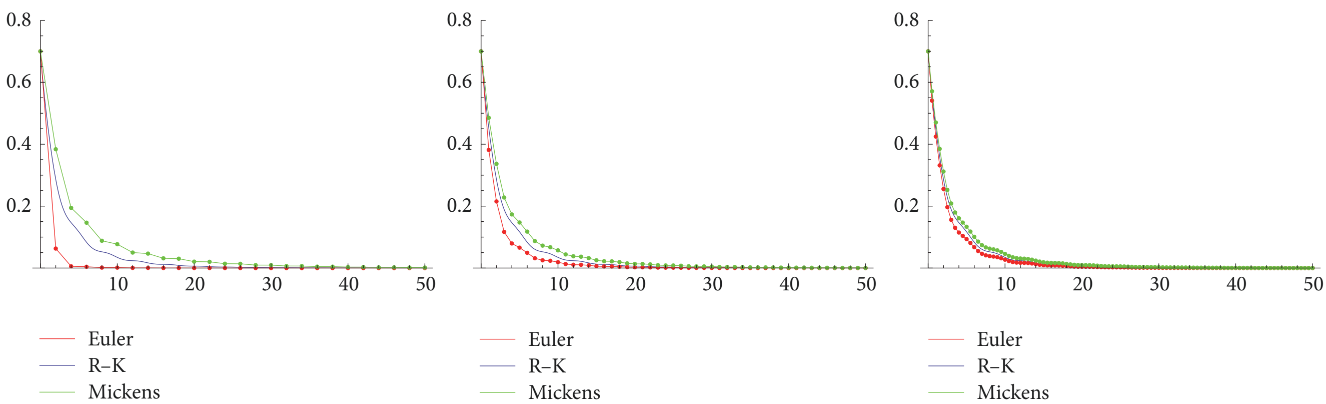

FIGURE 4: SI; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

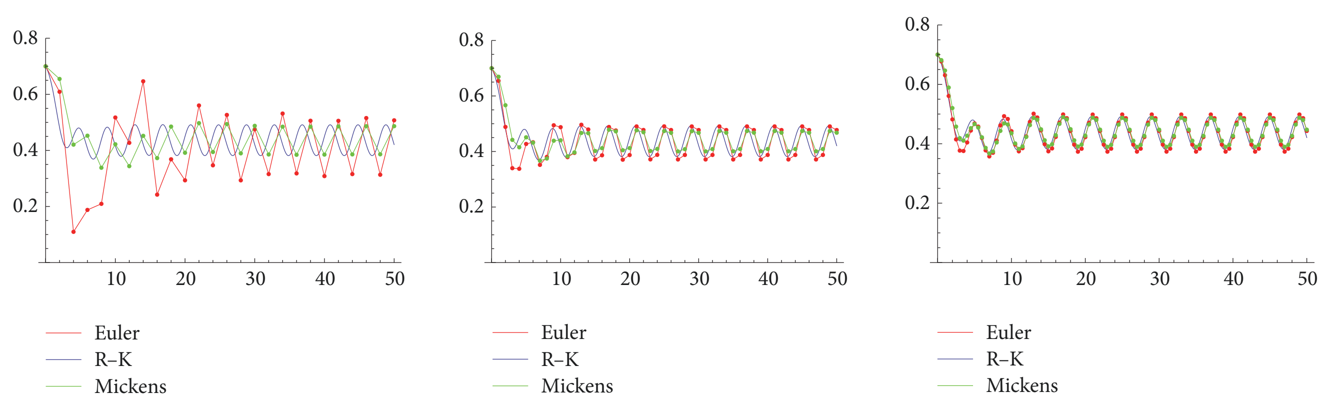

FIGURE 5: SI; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

Next, we change our incidence function and consider $\phi(S, I) = SI/(1 + 0.7I)$, maintaining the set of parameters. Letting $b = 0.3$ we have extinction for the continuous model and letting $b = 0.9$ we have persistence for the continuous-time model. Note that the thresholds $\mathscr{R}_C^u$, $\mathscr{R}_C^\ell$, $\mathscr{R}_D^u$, and $\mathscr{R}_D^\ell$ are similar to the mass action case. Figures 6 and 7 illustrate this situation.

Doing corresponding simulations and comparisons for our model with $\phi(h) = (1 - e^{-0.002h})/(0.002)$ instead of $\phi(h) = h + 0.2h^2$, we can draw the same conclusions regarding extinction/persistence, relation to Zhang's model and the model obtained by Euler method.

5.2. Simulation with Real Data. In this subsection, we present some simulation regarding measles. This disease is endemic in some countries such as France. In that country, with the measles outbreak in 2011, a vaccination policy that lowered the number of reported cases was introduced. We will focus on measles in France, between 2012 and 2016. For a study concerning the period before 2012, see [36]. For

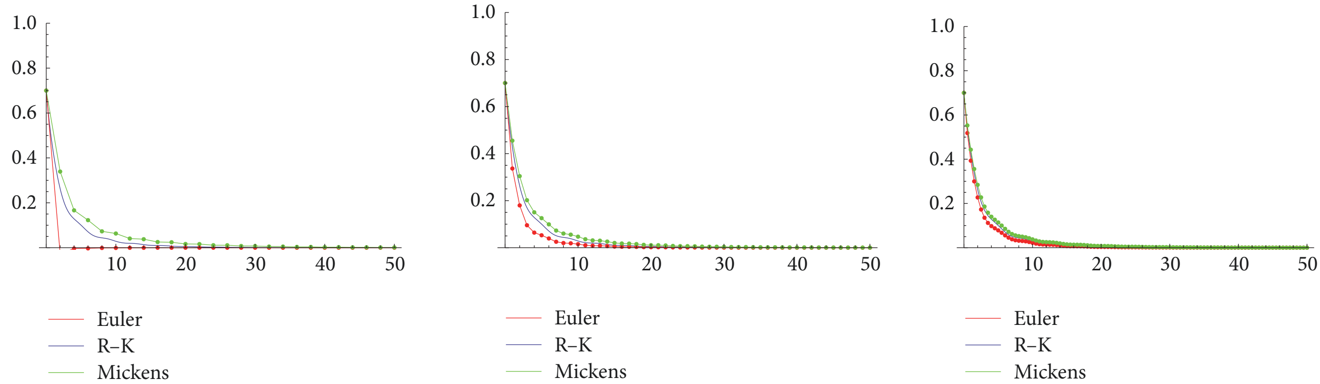

Figure 6: $SI/(1+0.7I)$; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

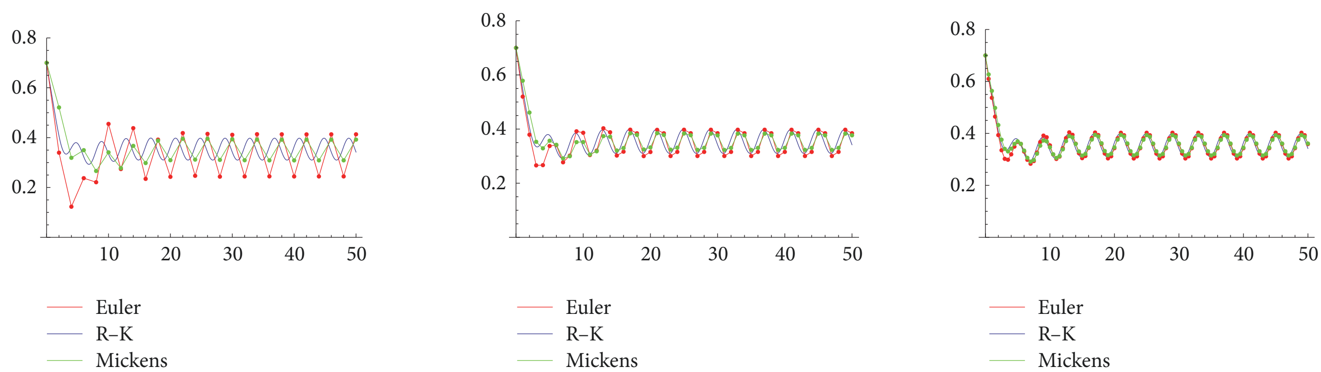

Figure 7: $SI/(1+0.7I)$; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

our parameters estimation, we gathered information from several websites. We considered standard incidence functions $\psi(V_{n+1}, I_n) = V_{n+1}I_n/P_n$ and $\varphi(S_{n+1}, I_n) = S_{n+1}I_n/P_n$, where P_n is the total population. Inspired in the time series for the infectives (https://ecdc.europa.eu), we considered $\sigma_n = 0.03$ and β_n given by

$$\beta_n = \begin{cases} 3.8 + 10 \sin\left(\dfrac{(n+1)\pi}{6}\right), & \text{if } \left\lfloor \dfrac{n}{12} \right\rfloor \leqslant 5 \\ 2.7, & \text{otherwise.} \end{cases} \quad (67)$$

The remaining parameters were considered time independent and were inspired in data contained in the websites http://www.worldbank.org, https://data.oecd.org, and http://www.geoba.se. Namely, we took the mortality rate $\mu_n = 0.0007$, the newborns $\Lambda_n = 50000$, the disease induced mortality $\alpha_n = 0.000375$, the immunity loss $\eta_n = 0.001$, the vaccination rate $p_n = 0.001$, and the recovery rate $\gamma_n = 0.957$. We used the initial conditions $S_0 = 7.20428 \times 10^6$, $I_0 = 106$, $V_0 = 5.84372 \times 10^7$, and $R_0 = 1.81918 \times 10^4$. In Figure 8, we plot the real data for the infectives and the output given by our model.

Can be seen, in a general way, that our model behaves in the same manner as the real data. It seems that if the vaccination policy in France continues to be very strict, it may decrease the number of cases.

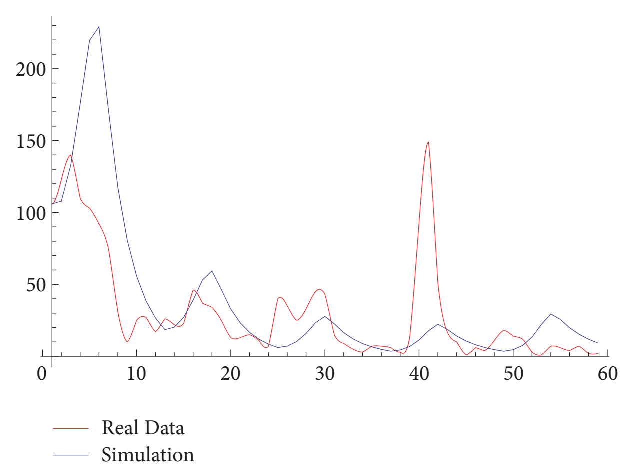

Figure 8: Measles (2012-2016), simulation.

6. Conclusions

We considered a discretization procedure, based on Mickens NSFD scheme, to get a discrete-time model from a continuous time with vaccination and incidence given by a general function. For a family of models containing the previous discrete-time model, we achieved results on the persistence and the extinction of the disease (Theorems 6 and 7). They contain the results of Zhang [20] as a particular case.

Our threshold conditions depend on the parameters of the model and of the incidence function derivative, with respect to the infectives, computed on some disease-free solution. This agrees with the continuous counterparts of these results [8].

We also considered the problem of establishing the consistency of the continuous-time model and the discrete-time model for small time steps, in the sense that if the time step is small enough when we have persistence (extinction) for the continuous-time model we also have persistence (extinction) for the discrete-time model (at least for situations where Theorems 1 and 2 allow us to conclude that we have persistence or extinction). Assuming the differentiability of parameters, our result on this direction, Theorem 9, furnishes an interval $[0, a]$, where a depends only on the parameters of the model and their derivatives, where there is consistency.

We present an example of a periodic system of period 1 where the continuous and the discrete-time system with time step $h = 1/L$ are not consistent. Namely, for that time step, we will have persistence for the continuous-time model and extinction for the discrete-time model. These examples show the importance of knowing that for time steps smaller than some explicit value we have consistency, a type of result like the one in Theorem 9.

Finally, we carried out some simulations to illustrate our results. As one might expect our simulations furnish evidence that we may have consistency in intervals whose lengths are several times bigger than the length of the given interval in Theorem 9. Additionally, we used our model to describe a real situation, namely, the case of measles incidence in France in the period 2012-2016, and compared our results with the real-time series for the infectives. We found in general that the predictive behaviour of our model is very similar to the real data.

We remark that this work is far from giving a complete answer to the problem of consistency between our discrete and continuous-time models. Moreover, in addition to a further discussion of the qualitative behaviour of the continuous and discrete systems, in future work it would be very interesting to discuss the convergence of the methods and to carry out some numerical analysis in our general nonautonomous setting.

Appendix

Proof of the Results in Section 3

We begin this section by noting a simple consequence of our assumptions that will be used several times throughout the proofs: it follows from (H3) that there are constants $K > 0$ and $\theta \in]0, 1[$ such that

$$\prod_{k=m}^{n-1} \frac{1}{1+\mu_k} < K\theta^{n-m}, \tag{A.1}$$

for $m, n \in \mathbb{N}$ sufficiently large. Additionally, using (H1), (H2), and (H5), we have

$$\begin{aligned} \frac{\varphi(x,y)}{y} &= \frac{\varphi(x,y)-\varphi(x,0)}{y-0} \leqslant \frac{\varphi(x,y)-\varphi(x,0)}{y-0} \\ &\leqslant \partial_2\varphi(x,0) = \left|\partial_2\varphi(x,0)-\partial_2\varphi(0,0)\right| \\ &\leqslant k_\varphi x \end{aligned} \tag{A.2}$$

and thus

$$\varphi(x,y) \leqslant k_\varphi xy. \tag{A.3}$$

Similarly

$$\psi(x,y) \leqslant k_\psi xy. \tag{A.4}$$

We will now proceed with the proof of the results in Section 3.

Proof of Lemma 4. Let $S_n > 0$, $I_n > 0$, $R_n > 0$, and $V_n > 0$. By (1), (A.3), and (A.4), we obtain

$$V_{n+1} = \frac{V_n + p_n S_{n+1}}{1+\mu_n+\eta_n+\sigma_n k_\psi I_n} \tag{A.5}$$

and thus

$$S_{n+1} \geqslant \frac{\eta_n\left(V_n+\Lambda_n+S_n\right)+\left(\Lambda_n+S_n\right)\left(1+\mu_n+\sigma_n k_\psi I_n\right)}{\left(1+\mu_n+\beta_n k_\varphi I_n\right)\left(1+\mu_n+\eta_n+\sigma_n k_\varphi I_n\right)+p_n\left(1+\mu_n+\sigma_n k_\psi I_n\right)}. \tag{A.6}$$

Therefore, we conclude that $S_{n+1} > 0$ and $V_{n+1} > 0$. By the second and third equations in (1) we obtain

$$\begin{aligned} I_{n+1} &\geqslant \frac{I_n}{1+\mu_n+\alpha_n+\gamma_n}, \\ R_{n+1} &= \frac{\gamma_n I_{n+1}+R_n}{1+\mu_n} \end{aligned} \tag{A.7}$$

and we conclude that $I_{n+1} > 0$ and $R_{n+1} > 0$. The previous inequalities allow us to conclude by induction that $S_n > 0$, $I_n > 0$, $R_n > 0$, and $V_n > 0$ for all $n \in \mathbb{N}$. In the same way, we can conclude that, if $S_0 \geqslant 0$, $I_0 \geqslant 0$, $R_0 \geqslant 0$ and $V_0 \geqslant 0$, then $S_n \geqslant 0$. $I_n \geqslant 0$, $R_n \geqslant 0$ and $V_n \geqslant 0$ for all $n \in \mathbb{N}$. This proves (i) and (ii) in Lemma 4.

By (1), we have

$$N_{n+1} \leqslant \frac{\Lambda_n}{1+\mu_n} + \frac{N_n}{1+\mu_n}, \tag{A.8}$$

where $N_n = S_n+I_n+R_n+V_n$ is the total population. By Lemma 2 in [21] we obtain the result. □

Proof of Lemma 5. To show that $\mathcal{R}_D^\ell(\xi^*, \lambda)$ is independent of the selection of $\xi^* = (x_n^*, y_n^*)$, a fixed solution of (12), it is important to note that according to (ii) in Lemma 3, for any $\varepsilon > 0$ and any solution $\xi = (x_n, y_n)$ of system (12) with initial value $x_0 > 0$, $y_0 > 0$, there exists an $N \in \mathbb{N}^+$ such that, for $k \geqslant N$, we have $|x_k - x_k^*| \leqslant \varepsilon$ and $|y_k - y_k^*| \leqslant \varepsilon$. Hence,

$$\begin{aligned} x_k^* - \varepsilon \leqslant x_k \leqslant x_k^* + \varepsilon \\ y_k^* - \varepsilon \leqslant y_k \leqslant y_k^* + \varepsilon. \end{aligned} \tag{A.9}$$

By (H1), we have

$$\begin{aligned} &\left|\partial_2\varphi(x_k, 0) - \partial_2\varphi(x_k^*, 0)\right| \leqslant k_\varphi \left|x_k - x_k^*\right| \leqslant k_\varphi \varepsilon, \\ &\left|\partial_2\psi(y_k, 0) - \partial_2\psi(y_k^*, 0)\right| \leqslant k_\psi \left|y_k - y_k^*\right| \leqslant k_\psi \varepsilon. \end{aligned} \tag{A.10}$$

So,

$$\begin{aligned} &\partial_2\varphi(x_k^*, 0) - k_\varphi\varepsilon \leqslant \partial_2\varphi(x_k, 0) \leqslant \partial_2\varphi(x_k^*, 0) + k_\varphi\varepsilon, \\ &\partial_2\psi(y_k^*, 0) - k_\psi\varepsilon \leqslant \partial_2\psi(y_k, 0) \\ &\qquad \leqslant \partial_2\psi(y_k^*, 0) + k_\psi\varepsilon. \end{aligned} \tag{A.11}$$

Combining the previous computations, we get

$$\begin{aligned} &\frac{1 + \beta_k\partial_2\varphi(x_{k+1}^*, 0) + \sigma_k\partial_2\psi(y_{k+1}^*, 0) - \overline{L}_k\varepsilon}{1 + \mu_k + \alpha_k + \gamma_k} \\ &\quad \leqslant \frac{1 + \beta_k\partial_2\varphi(x_{k+1}, 0) + \sigma_k\partial_2\psi(y_{k+1}, 0)}{1 + \mu_k + \alpha_k + \gamma_k} \\ &\quad \leqslant \frac{1 + \beta_k\partial_2\varphi(x_{k+1}^*, 0) + \sigma_k\partial_2\psi(y_{k+1}^*, 0) + \overline{L}_k\varepsilon}{1 + \mu_k + \alpha_k + \gamma_k}, \end{aligned} \tag{A.12}$$

where $\overline{L}_k = \beta_k k_\varphi + \sigma_k k_\psi$.

Let

$$r_k = \frac{1 + \beta_k\partial_2\varphi(x_{k+1}^*, 0) + \sigma_k\partial_2\psi(y_{k+1}^*, 0)}{1 + \mu_k + \alpha_k + \gamma_k}. \tag{A.13}$$

Using (H1) and (iv) in Lemma 3, it is easy to see that

$$r_k \leqslant \frac{1 + 2\beta^u k_\varphi M + 2\sigma^u k_\psi M}{1 + \mu^l + \alpha^l + \gamma^l} =: r, \tag{A.14}$$

for sufficiently large $k \in \mathbb{N}$, and that

$$\overline{L} = \frac{\overline{L}_k}{1 + \mu_k + \alpha_k + \gamma_k} \leqslant \frac{\beta^u k_\varphi + \sigma^u k_\psi}{1 + \mu^l + \alpha^l + \gamma^l} =: C. \tag{A.15}$$

So, for sufficiently large n,

$$\begin{aligned} \prod_{k=n}^{n+\lambda}\left(r_k + \frac{\overline{L}_k\varepsilon}{1 + \mu_k + \alpha_k + \gamma_k}\right) &\leqslant \prod_{k=n}^{n+\lambda}(r_k + C\varepsilon) \\ &= \prod_{k=n}^{n+\lambda} r_k + \Theta_\varepsilon \end{aligned} \tag{A.16}$$

where

$$\begin{aligned} \Theta_\varepsilon = \binom{\lambda+1}{\lambda} r^\lambda C\varepsilon + \dots + \binom{\lambda+1}{1} r C^\lambda \varepsilon^\lambda \\ + C^{\lambda+1}\varepsilon^{\lambda+1}. \end{aligned} \tag{A.17}$$

Analogously

$$\prod_{k=n}^{n+\lambda}\left(r_k - \frac{\overline{L}_k\varepsilon}{1 + \mu_k + \alpha_k + \gamma_k}\right) \geqslant \prod_{k=n}^{n+\lambda} r_k - \Theta_\varepsilon. \tag{A.18}$$

By (A.12), we obtain

$$\begin{aligned} -\Theta_\varepsilon + \liminf_{n\to+\infty} \prod_{k=n}^{n+\lambda} r_k &\leqslant \mathcal{R}_D^\ell(\xi^*, \lambda) \\ &\leqslant \Theta_\varepsilon + \liminf_{n\to+\infty} \prod_{k=n}^{n+\lambda} r_k. \end{aligned} \tag{A.19}$$

Thus,

$$\left|\mathcal{R}_D^\ell(\xi^*, \lambda) - \mathcal{R}_D^\ell(\xi, \lambda)\right| < \Theta_\varepsilon \tag{A.20}$$

and, by the arbitrariness of ε, we obtain $\mathcal{R}_D^\ell(\xi, \lambda) = \mathcal{R}_D^\ell(\xi^*, \lambda)$. Replacing lim inf by lim sup in the preceding argument, we reach a similar conclusion for $\mathcal{R}_D^u(\xi^*, \lambda)$. The result follows. □

Proof of Theorem 6. First note that the original system (1) can be rewritten as follows:

$$\begin{aligned} S_{n+1} &= \frac{1}{1 + \mu_n + p_n}\left(\Lambda_n + S_n - \beta_n\varphi(S_{n+1}, I_n) + \eta_n V_{n+1}\right) \\ I_{n+1} &= \frac{1}{1 + \mu_n + \alpha_n + \gamma_n}\left(\beta_n\varphi(S_{n+1}, I_n) + \sigma_n\psi(V_{n+1}, I_n) + I_n\right) \\ R_{n+1} &= \frac{1}{1 + \mu_n}\left(\gamma_n I_{n+1} + R_n\right) \\ V_{n+1} &= \frac{1}{1 + \mu_n + \eta_n}\left(p_n S_{n+1} - \sigma_n\psi(V_{n+1}, I_n) + V_n\right), \end{aligned} \tag{A.21}$$

$n = 0, 1, \dots$.

Firstly, we will establish (a). Since $\mathcal{R}_D^u(\lambda) < 1$, we can choose $\varepsilon_0 > 0$, $\varepsilon \in]0, 1[$ and a sufficiently large integer $N_1 \in \mathbb{N}$ such that

$$\prod_{k=n}^{n+\lambda} \frac{1 + \beta_k\partial_2\varphi(x_{k+1}, 0) + \sigma_k\partial_2\psi(y_{k+1}, 0) + \left(\beta^u k_\varphi + \sigma^u k_\psi\right)\varepsilon_0}{1 + \mu_k + \alpha_k + \gamma_k} < \varepsilon \tag{A.22}$$

for all $n \geqslant N_1$.

For any solution (S_n, I_n, R_n, V_n) of (A.21) with initial conditions $S_0 > 0$, $I_0 > 0$, $R_0 > 0$ and $V_0 > 0$, we have

$$S_{n+1} \leqslant \frac{\Lambda_n + \eta_n V_{n+1} + S_n}{1 + \mu_n + p_n}$$
$$V_{n+1} \leqslant \frac{p_n S_{n+1} + V_n}{1 + \mu_n + \eta_n}. \tag{A.23}$$

By the comparison principle, we obtain $S_n \leqslant x_n$ and $V_n \leqslant y_n$ for all $n \in \mathbb{N}$, where (x_n, y_n) is the solution of (12) with initial condition $(x_0, y_0) = (S_0, V_0)$. According to Lemma 3, the solution (x_n^*, y_n^*) is globally uniformly attractive and thus, for the aforementioned $\varepsilon_0 > 0$, there exists an $N_2 \in \mathbb{N}$ such that

$$\left|x_n - x_n^*\right| \leqslant \varepsilon_0,$$
$$\left|y_n - y_n^*\right| \leqslant \varepsilon_0 \qquad \forall n \geqslant N_2 \tag{A.24}$$

From this, it may be concluded that

$$S_n \leqslant x_n^* + \varepsilon_0,$$
$$V_n \leqslant y_n^* + \varepsilon_0 \qquad \forall n \geqslant N_2. \tag{A.25}$$

By the second equation of (1) we get

$$\begin{aligned} I_{n+1} &= \frac{1}{1 + \mu_n + \alpha_n + \gamma_n} \left(\beta_n \varphi\left(S_{n+1}, I_n\right) + \sigma_n \psi\left(V_{n+1}, I_n\right) + I_n\right) \\ &= \frac{1}{1 + \mu_n + \alpha_n + \gamma_n} \left(\beta_n \frac{\varphi\left(S_{n+1}, I_n\right)}{I_n} + \sigma_n \frac{\psi\left(V_{n+1}, I_n\right)}{I_n} + 1\right) I_n. \end{aligned} \tag{A.26}$$

By (H5), we have

$$\frac{\varphi\left(S_{n+1}, I_n\right)}{I_n} \leqslant \partial_2 \varphi\left(S_{n+1}, 0\right),$$
$$\frac{\psi\left(V_{n+1}, I_n\right)}{I_n} \leqslant \partial_2 \psi\left(V_{n+1}, 0\right). \tag{A.27}$$

By (H1), $x \mapsto \partial_2\varphi(x, 0)$ and $x \mapsto \partial_2\psi(x, 0)$ are nondecreasing and also Lipschitz, so, using (A.25) we obtain

$$\begin{aligned} &\partial_2 \varphi\left(S_{n+1}, 0\right) - \partial_2 \varphi\left(x_{n+1}^*, 0\right) \\ &\quad \leqslant \partial_2 \varphi\left(x_{n+1}^* + \varepsilon_0, 0\right) - \partial_2 \varphi\left(x_{n+1}^*, 0\right) \\ &\quad = \left|\partial_2 \varphi\left(x_{n+1}^* + \varepsilon_0, 0\right) - \partial_2 \varphi\left(x_{n+1}^*, 0\right)\right| \leqslant k_\varphi \varepsilon_0 \end{aligned} \tag{A.28}$$

and thus

$$\partial_2 \varphi\left(S_{n+1}, 0\right) \leqslant \partial_2 \varphi\left(x_{n+1}^*, 0\right) + k_\varphi \varepsilon_0. \tag{A.29}$$

Analogously

$$\partial_2 \psi\left(V_{n+1}, 0\right) \leqslant \partial_2 \psi\left(y_{n+1}^*, 0\right) + k_\psi \varepsilon_0. \tag{A.30}$$

Therefore, by (A.22), (A.26), (A.27), (A.29), and (A.30) we have

$$\begin{aligned} &I_{n+1} \\ &\leqslant \frac{\beta_n\left(\partial_2 \varphi\left(x_{n+1}^*, 0\right) + k_\varphi \varepsilon_0\right) + \sigma_n\left(\partial_2 \psi\left(y_{n+1}^*, 0\right) + k_\psi \varepsilon_0\right) + 1}{1 + \mu_n + \alpha_n + \gamma_n} I_n \\ &\leqslant \varepsilon I_n, \end{aligned} \tag{A.31}$$

for all $n \geqslant N_2$. We conclude that $I_m \leqslant \varepsilon^{m-N_2} I_{N_2} \to 0$ as $m \to \infty$. This completes the proof of (a).

Next, to establish (b), let us consider two arbitrary solutions of the original system $(S_n^{(1)}, I_n^{(1)}, V_n^{(1)}, R_n^{(1)})$ and $(S_n^{(2)}, I_n^{(2)}, V_n^{(2)}, R_n^{(2)})$ and λ a constant such that $R_0^u(\lambda) < 1$. Let $\iota_n = I_n^{(1)} - I_n^{(2)}$ and $\rho_n = R_n^{(1)} - R_n^{(2)}$. By (9), we have

$$\begin{aligned} \rho_{n+1} - \rho_n &= \left(R_{n+1}^{(1)} - R_n^{(1)}\right) - \left(R_{n+1}^{(2)} - R_n^{(2)}\right) \\ &= \gamma_n\left(I_{n+1}^{(1)} - I_{n+1}^{(2)}\right) - \mu_n\left(R_{n+1}^{(1)} - R_{n+1}^{(2)}\right) \\ &= \gamma_n \iota_{n+1} - \mu_n \rho_{n+1}. \end{aligned} \tag{A.32}$$

Because $R_0^u(\lambda) < 1$, we conclude that $\iota_n \to 0$ as $n \to +\infty$ and therefore, given $\varepsilon > 0$, there is $N \in \mathbb{N}$ sufficiently large such that, for $n \geqslant N$,

$$\rho_{n+1} + \mu_n \rho_{n+1} = \gamma_n \iota_{n+1} + \rho_n < \varepsilon + \rho_n \tag{A.33}$$

Thus, since $\mu_n > 0$, we get

$$\rho_{n+1} < \frac{\varepsilon}{1 + \mu_n} + \frac{\rho_n}{1 + \mu_n}. \tag{A.34}$$

and proceeding by induction

$$\rho_{n+1} < \left(\prod_{m=0}^{n-1} \frac{1}{1 + \mu_m}\right) \rho_0 + \varepsilon \sum_{m=0}^{n-1} \left(\prod_{k=m}^{n-1} \frac{1}{1 + \mu_k}\right). \tag{A.35}$$

By (H3) and (A.1), we conclude that

$$\limsup_{n \to +\infty} \rho_n = 0. \tag{A.36}$$

Thus, $\rho_n = R_n^{(1)} - R_n^{(2)} \to 0$ as $n \to +\infty$. Similar computations show that $S_n^{(1)} - S_n^{(2)} \to 0$ and $V_n^{(1)} - V_n^{(2)} \to 0$. This proves (b) and the result follows. □

Proof of Theorem 7. Since $\mathscr{R}_0^\ell(\lambda) > 1$, there are $\varepsilon, \varepsilon_0 > 0$ such that

$$\begin{aligned} &\liminf_{n \to +\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \partial_2 \varphi\left(x_{k+1}^*, 0\right) + \sigma_k \partial_2 \psi\left(y_{k+1}^*, 0\right) - u}{1 + \mu_k + \alpha_k + \gamma_k} \\ &\quad > 1 + \varepsilon, \end{aligned} \tag{A.37}$$

for all $u \in [0, \varepsilon_0]$.

We claim that there is $\varepsilon_1 > 0$ such that

$$\limsup_{n\to\infty} I_n > \varepsilon_1 \tag{A.38}$$

for every solution with positive initial conditions of system (1).

We proceed by contradiction. Assume that (A.38) does not hold. Then, for each ε_1 there is $N_1 \in \mathbb{N}$ and a solution (S_n, I_n, R_n, V_n) with positive initial conditions such that $I_n \leqslant \varepsilon_1$ for all $n \geqslant N_1$. By (iii) in Lemma 4, we can assume that S_n, $V_n < M$ for all $n \geqslant N_1$. By (A.3) we have

$$\varphi(S_{n+1}, I_n) = k_\varphi S_{n+1} I_n \leqslant k_\varphi M \varepsilon_1 \tag{A.39}$$

and likewise, by (A.4), we get

$$\psi(V_{n+1}, I_n) = k_\psi V_{n+1} I_n \leqslant k_\psi M \varepsilon_1. \tag{A.40}$$

By (1), (A.39), and (A.40), we have

$$\begin{aligned} S_{n+1} &\geqslant \frac{\Lambda_n + \eta_n V_{n+1} + S_n - \beta^u k_\varphi M \varepsilon_1}{1 + \mu_n + p_n} \\ V_{n+1} &\geqslant \frac{p_n S_{n+1} + V_n - \sigma^u k_\psi M \varepsilon_1}{1 + \mu_n + \eta_n}. \end{aligned} \tag{A.41}$$

for all $n \geqslant N_1$.

Given $\varepsilon_1 > 0$, consider the auxiliary system

$$\begin{aligned} x_{n+1} &= \frac{\Lambda_n + \eta_n y_{n+1} + x_n - \beta^u k_\varphi M \varepsilon_1}{1 + \mu_n + p_n} \\ y_{n+1} &= \frac{p_n x_{n+1} + y_n - \sigma^u k_\psi M \varepsilon_1}{1 + \mu_n + \eta_n}. \end{aligned} \tag{A.42}$$

For any $n_0 \in \mathbb{N}$ and $x_0, y_0 \in \mathbb{R}^+$, let (x_n, y_n) be the solution of (12) with initial condition $(x_{n_0}, y_{n_0}) = (x_0, y_0)$ and let $(\overline{x}_{n_0}, \overline{y}_{n_0}) = (x_0, y_0)$ be the solution of (A.42) with the same initial condition. By (iii) in Lemma 3 we obtain

$$\begin{aligned} &\sup_{n\in\mathbb{N}_0} \{|\overline{x}_n - x_n| + |\overline{y}_n - y_n|\} \\ &\quad \leqslant LM\left(\beta^u k_\varphi + \sigma^u k_\psi\right)\varepsilon_1 \end{aligned} \tag{A.43}$$

and thus we can take $\varepsilon_1 > 0$ small enough such that

$$\sup_{n\in\mathbb{N}_0} \{|\overline{x}_n - x_n| + |\overline{y}_n - y_n|\} \leqslant \frac{\varepsilon_0}{2}. \tag{A.44}$$

On the other hand, by (ii) in Lemma 3, there is $N_2 \geqslant N_1$ sufficiently large such that

$$|x_n^* - x_n| + |y_n^* - y_n| \leqslant \frac{\varepsilon_0}{2}, \tag{A.45}$$

for all $n \geqslant N_2$. Therefore,

$$|\overline{x}_n - x_n^*| + |\overline{y}_n - y_n^*| \leqslant \varepsilon_0, \tag{A.46}$$

for all $n \geqslant N_2$.

Noting that (A.41) can be written as

$$\begin{aligned} &S_{n+1} \\ &\geqslant \frac{\eta_n p_n (1+\mu_n)(1+\mu_n+p_n+\eta_n)\left(\Lambda_n + S_n - \beta^u M k_\varphi \varepsilon_1\right)}{(1+\mu_n)(1+\mu_n+p_n)(1+\mu_n+p_n+\eta_n)} \\ &\quad + \frac{\left(V_n - \sigma^u M k_\psi \varepsilon_1\right)(1+\mu_n+p_n)}{(1+\mu_n)(1+\mu_n+p_n)(1+\mu_n+p_n+\eta_n)} \\ &V_{n+1} \\ &\geqslant \frac{p_n\left(\Lambda_n + S_n - \beta^u M k_\varphi \varepsilon_1\right) + \left(V_n - \sigma^u M k_\psi \varepsilon_1\right)(1+\mu_n+p_n)}{(1+\mu_n)(1+\mu_n+p_n+\eta_n)} \end{aligned} \tag{A.47}$$

and, using (A.42) and (A.46), we conclude that

$$\begin{aligned} S_n &\geqslant \overline{x}_n \geqslant x_n^* - \varepsilon_0, \\ V_n &\geqslant \overline{y}_n \geqslant y_n^* - \varepsilon_0, \end{aligned} \tag{A.48}$$

for all $n \geqslant N_2$. Thus, since $S_n \leqslant x_n^*$ and $V_n \leqslant y_n^*$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} |S_n - x_n^*| &\leqslant \varepsilon_0, \\ |V_n - y_n^*| &\leqslant \varepsilon_0, \end{aligned} \tag{A.49}$$

for all $n \geqslant N_2$. By (H1), we have, for all $n \geqslant N_2$,

$$\begin{aligned} &\left|\partial_2\varphi(S_{k+1}, 0) - \partial_2\varphi(x_{k+1}^*, 0)\right| \leqslant k_\varphi \left|S_{k+1} - x_{k+1}^*\right| \\ &\quad \leqslant k_\varphi \varepsilon_0 \end{aligned} \tag{A.50}$$

and thus

$$\begin{aligned} \partial_2\varphi(x_{k+1}^*, 0) - k_\varphi\varepsilon_0 &\leqslant \partial_2\varphi(S_{k+1}, 0) \\ &\leqslant \partial_2\varphi(x_{k+1}^*, 0) + k_\varphi\varepsilon_0. \end{aligned} \tag{A.51}$$

Reasoning similarly we obtain, for all $n \geqslant N_2$,

$$\begin{aligned} \partial_2\psi(y_{k+1}^*, 0) - k_\psi\varepsilon_0 &\leqslant \partial_2\psi(V_{k+1}, 0) \\ &\leqslant \partial_2\psi(y_{k+1}^*, 0) + k_\psi\varepsilon_0. \end{aligned} \tag{A.52}$$

By (H1) and (H2), we conclude that

$$\begin{aligned} \varphi(S_{n+1}, I_n) &= \varphi(S_{n+1}, 0) + \partial_2\varphi(S_{n+1}, \varepsilon_n)(I_n - 0) \\ &= \partial_2\varphi(S_{n+1}, \varepsilon_n) I_n, \end{aligned} \tag{A.53}$$

for some $\varepsilon_n \in [0, \varepsilon_1]$, and all $n \geqslant N_2$. Thus, by continuity of $\partial_2\varphi$,

$$\begin{aligned} \frac{\varphi(S_{n+1}, I_n)}{I_n} &= \partial_2\varphi(S_{n+1}, \varepsilon_n) \\ &\geqslant \partial_2\varphi(S_{n+1}, 0) - \theta_1(\varepsilon_1), \end{aligned} \tag{A.54}$$

with $\theta_1(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$, for all $n \geqslant N_2$. Thus, by (A.51)

$$\frac{\varphi(S_{n+1}, I_n)}{I_n} \geqslant \partial_2\varphi(x_{k+1}^*, 0) - k_\varphi\varepsilon_0 - \theta_1(\varepsilon_1), \tag{A.55}$$

where $\theta_1(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. Similarly, for all $n \geqslant N_2$, we have, by (A.52)

$$\frac{\psi(V_{n+1}, I_n)}{I_n} \geqslant \partial_2\psi(y^*_{k+1}, 0) - k_\psi\varepsilon_0 - \theta_2(\varepsilon_1), \quad \text{(A.56)}$$

where $\theta_2(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. From the second equation in (1), we have

$$\begin{aligned} I_{n+1} &= \frac{1 + \beta_n\varphi(S_{n+1}, I_n)/I_n + \sigma_n\psi(V_{n+1}, I_n)/I_n}{1 + \mu_n + \alpha_n + \gamma_n} I_n \\ &\geqslant \frac{1 + \beta_n\partial_2\varphi(x^*_{k+1}, 0) + \sigma_n\partial_2\psi(y^*_{k+1}, 0) - u}{1 + \mu_n + \alpha_n + \gamma_n} I_n, \end{aligned} \quad \text{(A.57)}$$

where

$$u = \beta^u\left(k_\varphi\varepsilon_0 + \theta_1(\varepsilon_1)\right) + \sigma^u\left(k_\psi\varepsilon_0 + \theta_2(\varepsilon_1)\right), \quad \text{(A.58)}$$

for all $n \geqslant N_2$. Letting ε_1 be sufficiently small, we conclude, according to (A.37) and (A.57), that $I_n \to +\infty$ as $n \to +\infty$, a contradiction. Thus, we conclude that (A.38) holds.

Next we will prove the permanence of the infectives. By (iii) in Lemma 4, it is only necessary to prove that there is an $\varepsilon_2 > 0$ such that, for any solution (S_n, I_n, R_n, V_n) of (1) with positive initial conditions, we have

$$\liminf_{n\to+\infty} I_n > \varepsilon_2. \quad \text{(A.59)}$$

Recall that, since $\mathscr{R}^\ell_0(\lambda) > 1$, there are $\varepsilon, \varepsilon_0 > 0$ such that (A.37) holds for all $u \in [0, \varepsilon_0]$.

If (A.59) does not hold, then, given $\varepsilon_0 > 0$, there must be a sequence of solutions of (1), $((S_{n,k}, I_{n,k}, R_{n,k}, V_{n,k})_{n\in\mathbb{N}})_{k\in\mathbb{N}}$, with initial conditions $(S_{0,k}, I_{0,k}, R_{0,k}, V_{0,k})$ such that

$$\liminf_{n\to+\infty} I_{n,k} < \frac{\varepsilon_0}{k^2}. \quad \text{(A.60)}$$

From (A.38), for each $k \in \mathbb{N}$, there must be two sequences $(s_{m,k})_{m\in\mathbb{N}}$ and $(t_{m,k})_{m\in\mathbb{N}}$ such that $s_{m,k} \to +\infty$ as $m \to +\infty$,

$$0 < s_{1,k} < t_{1,k} < s_{2,k} < t_{2,k} < \cdots < s_{m,k} < t_{m,k} < \ldots, \quad \text{(A.61)}$$

$$I_{s_{m,k},k} > \frac{\varepsilon_0}{k}, \qquad I_{t_{m,k},k} < \frac{\varepsilon_0}{k^2}, \quad \text{(A.62)}$$

$$\frac{\varepsilon_0}{k^2} \leqslant I_{n,k} \leqslant \frac{\varepsilon_0}{k}, \quad \forall n \in [s_{m,k} + 1, t_{n,k} - 1] \cup \mathbb{N}. \quad \text{(A.63)}$$

Given $n \in [s_{m,k}, t_{m,k} - 1] \cup \mathbb{N}$, we have

$$\begin{aligned} I_{n+1,k} &= \frac{1 + \beta_n\varphi(S_{n+1,k}, I_{n,k})/I_{n,k} + \sigma_n\psi(V_{n+1,k}, I_{n,k})/I_{n,k}}{1 + \mu_n + \alpha_n + \gamma_n} \\ &\cdot I_{n,k} \geqslant \frac{1}{1 + \mu^u + \alpha^u + \gamma^u} I_{n,k} \end{aligned} \quad \text{(A.64)}$$

and therefore

$$\frac{\varepsilon_0}{k^2} > I_{t_{n,k},k} \geqslant \sigma^{t_{n,k}-s_{n,k}} I_{s_{n,k},k} > \frac{\sigma^{t_{n,k}-s_{n,k}}\varepsilon_0}{k}, \quad \text{(A.65)}$$

where

$$\sigma = \frac{1}{1 + \mu^u + \alpha^u + \gamma^u}. \quad \text{(A.66)}$$

Thus, by (A.62) and (A.65),

$$t_{n,k} - s_{n,k} \geqslant \frac{\ln k}{\ln(1/\sigma)} \longrightarrow +\infty \quad \text{as } k \longrightarrow +\infty. \quad \text{(A.67)}$$

In view of (A.67), we can choose $k_0 \in \mathbb{N}$ such that

$$t_{n,k} - s_{n,k} > N + \lambda + 1, \quad \text{(A.68)}$$

for all $k \geqslant k_0$.

Letting m and $k \geqslant k_0$ be sufficiently large and $\varepsilon_2 > 0$ be sufficiently small, we may assume that

$$\begin{aligned} S_{n,k} &\geqslant \frac{\Lambda_n + \eta_n V_{n,k} + S_{n,k} - \beta^u k_\varphi M\varepsilon_2}{1 + \mu_n + p_n} \\ V_{n,k} &\geqslant \frac{p_n S_{n,k} + V_{n,k} - \sigma^u k_\psi M\varepsilon_2}{1 + \mu_n + \eta_n} \end{aligned} \quad \text{(A.69)}$$

holds for all $n \in [s_{m,k} + 1, t_{m,k} - 1] \cap \mathbb{N}$.

Let $(\overline{x}_n, \overline{y}_n)$ be a solution of (A.42) with initial value $(\overline{x}_{s_{m,k}}, \overline{y}_{s_{m,k}}) = (S_{s_{m,k}}, V_{s_{m,k}})$. We have $S_n \geqslant \overline{x}_n$ and $V_n \geqslant \overline{y}_n$ for all $n \in [s_{m,k}, t_{m,k}] \cap \mathbb{N}$. Letting $\varepsilon_2 > 0$ in (A.69) be sufficiently small and (x_n, y_n) be the solution of (12) with $x_{s_{n,k}+1} = S_{s_{n,k}+1}$ and $y_{s_{n,k}} = V_{s_{n,k}}$, we have by (iii) in Lemma 3

$$|x_n - \overline{x}_n| + |y_n - \overline{y}_n| \leqslant \frac{\varepsilon_0}{2} \quad \text{(A.70)}$$

for all $n \in [s_{m,k}, t_{m,k}] \cap \mathbb{N}$. We conclude that

$$\begin{aligned} S_n &\geqslant \overline{x}_n \geqslant x_n - \frac{\varepsilon_0}{2} > x^*_n - \varepsilon_0, \\ V_n &\geqslant \overline{y}_n \geqslant y_n - \frac{\varepsilon_0}{2} > y^*_n - \varepsilon_0 \end{aligned} \quad \text{(A.71)}$$

for all $n \in [s_{m,k}, t_{m,k}] \cap \mathbb{N}$.

Proceeding like before, we obtain (A.51) and (A.52) with S_k and V_k replaced by $S_{n+1,k}$ and $V_{n+1,k}$, respectively, and, for sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} \frac{\varphi(S_{n+1,k}, I_{n,k})}{I_{n,k}} &\geqslant \partial_2\varphi(x^*_{k+1}, 0) - k_\varphi\varepsilon_0 - \theta_1(\varepsilon_1), \\ \frac{\psi(V_{n+1,k}, I_{n,k})}{I_{n,k}} &\geqslant \partial_2\psi(y^*_{k+1}, 0) - k_\psi\varepsilon_0 - \theta_2(\varepsilon_1), \end{aligned} \quad \text{(A.72)}$$

where $\theta_1(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. Therefore,

$$\begin{aligned} I_{n+1,k} &\geqslant \frac{1 + \beta_n\partial_2\varphi(x^*_{n+1}, 0) + \sigma_n\partial_2\psi(y^*_{n+1}, 0) - u}{1 + \mu_n + \alpha_n + \gamma_n} I_{n,k}, \end{aligned} \quad \text{(A.73)}$$

for all $n \in [s_{m,k}, t_{m,k}] \cap \mathbb{N}$, where u is given by (A.58). Thus,

$$\begin{aligned} \frac{\varepsilon_0}{k^2} &> I_{t_{m,k},k} \\ &\geqslant I_{t_{m,k}-\lambda,k} \prod_{n=t_{m,k}-\lambda}^{t_{m,k}} \frac{1+\beta_n \partial_2 \varphi(x_{n+1}^*,0)+\sigma_n \partial_n \psi(y_{n+1}^*,0)-u}{1+\mu_n+\alpha_n+\gamma_n} \\ &> \frac{\varepsilon_0}{k^2} \end{aligned} \quad \text{(A.74)}$$

which is a contradiction. The theorem follows. □

Data Availability

The real data used to undertake the simulation presented in the article is available in the websites: http://www.worldbank.org, https://data.oecd.org, and http://www.geoba.se.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Joaquim Mateus, César M. Silva, and S. Vaz were partially supported by FCT through CMA-UBI (Project UID/MAT/00212/2013).

References

[1] O. Diekmann and J. A. P. Heesterbeek, *Mathematical Epidemiology of Infectious Diseases, Model Building, Analysis and Interpretation*, John Wiley & Sons, 2000.

[2] W. Tan, J. Gao, and W. Fan, "Bifurcation analysis and chaos control in a discrete epidemic system," *Discrete Dynamics in Nature and Society*, Art. ID 974868, 13 pages, 2015.

[3] Y. Wang, Z. Teng, and M. Rehim, "Lyapunov functions for a class of discrete SIRS epidemic models with nonlinear incidence rate and varying population sizes," *Discrete Dynamics in Nature and Society*, Article ID 472746, 10 pages, 2014.

[4] Z. Wei and M. Le, "Existence and convergence of the positive solutions of a discrete epidemic model," *Discrete Dynamics in Nature and Society*, Art. ID 434537, 10 pages, 2015.

[5] J. Liu, B. Peng, and T. Zhang, "Effect of discretization on dynamical behavior of SEIR and SIR models with nonlinear incidence," *Applied Mathematics Letters*, vol. 39, pp. 60–66, 2015.

[6] Z. Hu, Z. Teng, and L. Zhang, "Stability and bifurcation analysis in a discrete SIR epidemic model," *Mathematics and Computers in Simulation*, vol. 97, pp. 80–93, 2014.

[7] Z. Hu, Z. Teng, C. Jia, C. Zhang, and L. Zhang, "Dynamical analysis and chaos control of a discrete SIS epidemic model," *Advances in Difference Equations*, 2014:58, 20 pages, 2014.

[8] E. Pereira, C. M. Silva, and J. A. da Silva, "A generalized nonautonomous SIRVS model," *Mathematical Methods in the Applied Sciences*, vol. 36, no. 3, pp. 275–289, 2013.

[9] Q. Cui, X. Yang, and Q. Zhang, "An NSFD scheme for a class of SIR epidemic models with vaccination and treatment," *Journal of Difference Equations and Applications*, vol. 20, no. 3, pp. 416–422, 2014.

[10] Q. Cui and Q. Zhang, "Global stability of a discrete SIR epidemic model with vaccination and treatment," *Journal of Difference Equations and Applications*, vol. 21, no. 2, pp. 111–117, 2015.

[11] D. Ding, Q. Ma, and X. Ding, "A non-standard finite difference scheme for an epidemic model with vaccination," *Journal of Difference Equations and Applications*, vol. 19, no. 2, pp. 179–190, 2013.

[12] D. Ding and X. Ding, "Dynamic consistent non-standard numerical scheme for a dengue disease transmission model," *Journal of Difference Equations and Applications*, vol. 20, no. 3, pp. 492–505, 2014.

[13] S. Jang and S. Elaydi, "Difference equations from discretization of a continuous epidemic model with immigration of infectives," *Canadian Applied Mathematics Quarterly*, vol. 11, no. 1, pp. 93–105, 2003.

[14] R. E. Mickens, "Discretizations of nonlinear differential equations using explicit nonstandard methods," *Journal of Computational and Applied Mathematics*, vol. 110, no. 1, pp. 181–185, 1999.

[15] R. E. Mickens and T. M. Washington, "A note on an NSFD scheme for a mathematical model of respiratory virus transmission," *Journal of Difference Equations and Applications*, vol. 18, no. 3, pp. 525–529, 2012.

[16] R. E. Mickens, "Nonstandard finite difference schemes for differential equations," *Journal of Difference Equations and Applications*, vol. 8, no. 9, pp. 823–847, 2002.

[17] C. Sun and W. Yang, "Global results for an SIRS model with vaccination and isolation," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 4223–4237, 2010.

[18] A. A. Khasnis and M. D. Nettleman, "Global warming and infectious disease," *Archives of Medical Research*, vol. 36, no. 6, pp. 689–696, 2005.

[19] P. E. Kloeden and C. Pötzsche, "Nonautonomous dynamical systems in the life sciences," in *Nonautonomous dynamical systems in the life sciences*, vol. 2102 of *Lecture Notes in Math.*, pp. 3–39, Springer, Cham, 2013.

[20] T. Zhang, "Permanence and extinction in a nonautonomous discrete SIRVS epidemic model with vaccination," *Applied Mathematics and Computation*, vol. 271, pp. 716–729, 2015.

[21] T. Zhang, Z. Teng, and S. Gao, "Threshold conditions for a non-autonomous epidemic model with vaccination," *Applicable Analysis: An International Journal*, vol. 87, no. 2, pp. 181–199, 2008.

[22] L. J. Allen, M. A. Jones, and C. F. Martin, "A discrete-time model with vaccination for a measles epidemic," *Mathematical Biosciences*, vol. 105, no. 1, pp. 111–131, 1991.

[23] L. J. S. Allen, "Some discrete-time *SI, SIR,* and *SIS*epidemic models," *Mathematical Biosciences*, vol. 124, no. 1, pp. 83–105, 1994.

[24] K. L. Cooke, "A discrete-time epidemic model with classes of infectives and susceptibles," *Theoretical Population Biology. An International Journal*, vol. 7, pp. 175–196, 1975.

[25] M. C. M. De Jong, O. Diekmann, and J. A. P. Heesterbeek, "The computation of R0 for discrete-time epidemic models with dynamic heterogeneity," *Mathematical Biosciences*, vol. 119, no. 1, pp. 97–114, 1994.

[26] K. Hattaf, A. A. Lashari, B. El Boukari, and N. Yousfi, "Effect of discretization on dynamical behavior in an epidemiological model," *Differential Equations and Dynamical Systems*, vol. 23, no. 4, pp. 403–413, 2015.

[27] V. Kaitala, M. Heino, and W. M. Getz, "Host-parasite dynamics and the evolution of host immunity and parasite fecundity

strategies," *Bulletin of Mathematical Biology*, vol. 59, no. 3, pp. 427–450, 1997.

[28] G. Di Lena and G. Serio, "A discrete method for the identification of parameters of a deterministic epidemic model," *Mathematical Biosciences*, vol. 60, no. 2, pp. 161–175, 1982.

[29] C. Lefevre and M.-P. Malice, "A discrete time model for an S-I-S Infectious disease with a random number of contacts between individuals," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 7, no. 5-8, pp. 785–792, 1986.

[30] C. Lefevre and P. Picard, "On the formulation of discrete-time epidemic models," *Mathematical Biosciences*, vol. 95, no. 1, pp. 27–35, 1989.

[31] J. Longini, "The generalized discrete-time epidemic model with immunity: a synthesis," *Mathematical Biosciences*, vol. 82, no. 1, pp. 19–41, 1986.

[32] R. E. Mickens, "A discrete-time model for the spread of periodic diseases without immunity," *BioSystems*, vol. 26, no. 3, pp. 193–198, 1992.

[33] C. C. Spicer, "The mathematical modeling of influenza," *British Medical Bulletin*, vol. 35, pp. 23–28, 1979.

[34] D. Pierre-Loti-Viaud, "Large deviations for discrete-time epidemic models," *Mathematical Biosciences*, vol. 117, no. 1-2, pp. 197–210, 1993.

[35] R. Webster West and J. R. Thompson, "Models for the simple epidemic," *Mathematical Biosciences*, vol. 141, no. 1, pp. 29–39, 1997.

[36] N. Bacar and E. H. Ait Dads, "On the probability of extinction in a periodic environment," *Journal of Mathematical Biology*, vol. 68, no. 3, pp. 533–548, 2014.